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EXPERIMENTAL BUILDING SCIENCE

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VOLUME TWO

BEING AN INTRODUCTION TO MECHANICS AND ITS APPLICATION IN THE DESIGN AND ERECTION OF BUILDINGS

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PREFACE

To the student of building construction, the most fascinating pages in the history of industrial and scientific development are those which tell of the growth of the science of Mechanics, of the invention and the production on a large scale of mild steel and of Portland cement, and of the extended use of this knowledge and of these materials in the design and erection of modern buildings in steel and in reinforced concrete.*

It is a remarkable fact, however, that the knowledge and experience so gained has not as yet had any substantial effect upon the older methods of building construction, which are still to a large extent conditioned by traditional usage, rather than by conformity to the more scientific methods adopted in some of the newer forms of construction. In addition, as recent experience has shown, the introduction of new building materials and of new methods of construction have not always brought those immediate benefits which might have been expected. This has no doubt in part been due to the fact that, while it has been relatively easy to oust the older, traditional and less well-advertised methods of building construction, it has proved to be much more difficult to establish—even in these times of rapid change and re-adjustment—a common tradition of design and technique in the new materials, and to replace the large bodies of technical and craft skill which have thus been rendered ineffective. It is largely because of this fact that, in spite of the progress which has been made, the difficulties and responsibilities of the architect, the builder and the structural specialist are probably greater to-day than they have ever been in modern times.

The results of such an experience suggest that the most solid and lasting achievements in the development of building practice will not be found in the exclusive use of any one material—a method which usually results in a wasteful technique, when the new material is used in details for which it is not suited—but in the skilful use of all those materials which, either by the test of

* For a more detailed statement see J. Leask Manson, "Factors in the Development of Structural Practice," *The Structural Engineer*, March, 1926.

time or by the more rapid scientific tests of the laboratory, have proved themselves to be worthy of adoption and retention. Such a solution, however, will hardly be possible until there is a more widespread acquaintance with the chief properties of each material and with the general principles upon which their use in construction is based.

Apart from problems concerning the architectural form and style of buildings and the decorative use of materials—problems which would obviously be out of place in any volume included in this series—the design and construction of any building should, if it be carried out on rational lines, be the outcome of a knowledge of the external forces to be resisted, of the internal stresses resulting therefrom, and of the mechanical properties and the permanence of the materials to be employed. In the main, any consideration of the lasting powers or permanence of building materials must depend upon the application of the principles of Physics and Chemistry. The structural use and the strength of building materials can, however, be largely determined by the application of the principles of Mechanics, and it has therefore appeared to the authors that, under the general title of *Experimental Building Science*, and following the first volume (which dealt with the application of general elementary science to building work), a volume might well be devoted to the development of the more elementary and relevant sections of Mechanics, including Elasticity and the Strength of Materials, and to the application of this knowledge to well-known forms of building construction, the manner of treatment being such as to make it of value not only to the young student of both general and specialised construction, but also to the more mature reader who, for practical purposes, desires a straightforward and not too academic treatment of these topics.

The volume has been divided into three main sections. While the contents of each section are described in the opening paragraph in each case, reference may be made here to the general plan which has been adopted and to certain features which the authors hope will prove to be of special interest to the readers.

Section I deals with the general principles governing the equilibrium of systems of forces and with the applications of these

principles to the analysis of the forces acting in framed structures. When using some of the more recently introduced types of hoisting and erecting plant, the modern builder is faced with some interesting problems in stability, and the opportunity has been taken to utilise these problems to form an interesting and elementary introduction to the consideration of systems of forces which act in more than one plane. In this particular form the latter portion of this work is believed to be new; the authors hope that it will prove to be instructive and of practical value.

Section II is mainly devoted to the development of the ordinary theory of bending. In building up this important subject very careful attention has been given to the order in which the theory is developed. Thus the forces acting in a loaded beam are first considered as separate forces, apart from the strains and stresses set up in the material of the beam; fundamental relations are thus established in a simple manner before the full theory is introduced. Similarly the somewhat complex relations between shear stress and bending stress in beams are dealt with immediately after an elementary consideration of complementary shear stresses. This order of treatment, which is reasonably logical, should somewhat reduce the difficulties sometimes experienced by the student in dealing with this subject for the first time.

In **Section III** the principles developed in the two preceding sections are applied to modern forms of building construction. While no attempt has been made to give exhaustive details of construction or to elaborate the technique of testing, the information included should be sufficient to render the treatment complete and effective, and many references have been included to more advanced text-books for those who desire to pursue their studies further and in greater detail. The principles of construction in steel and in reinforced concrete, and especially the theory of column design, have thus only been developed far enough to be of service to the young student and to the general reader, but the treatment is such as to prepare the way for a more specialised study in any particular direction at a later stage should it be desired. In order to show how the older methods of construction may be modified to conform to more scientific methods of design, the subjects of Masonry and of Timber Construction have been

dealt with in greater detail than is usual except in works specially devoted to these methods of construction.

This volume assumes that the reader possesses a knowledge of the elementary principles of Mechanics (such, e.g., as are outlined in Chaps. VI and VII in Vol. I).

While there has been no intention of shirking the difficulties proper to the development of the subject, difficulties of a purely mathematical nature have been omitted where they add little to the elucidation of the subject. In cases where the more advanced methods of mathematical reasoning may be used with advantage by readers who are familiar with them, they have been included as alternative methods.

While graphical methods have been freely used throughout the volume, the authors lean to the view that arithmetical and algebraic methods of solution should be utilised in the first attempts since, as experience shows, a sounder knowledge of the principles involved can be obtained in this way. For this reason many worked Examples have been included.

All important statements of principles and all important formulae and expressions are, for ease of reference, *numbered consecutively through each chapter* (the former in large and the latter in small Roman numerals). The authors are of the opinion, however, that the memorising of these expressions and formulae will usually be of much less value than the possession of a clear conception of the arguments which lead up to the final results embodied in these expressions.

While the volume can be read and understood by a reader who is unable to conduct the experiments which are described in the text, the greatest profit will be derived if, as has been suggested herein by the authors, all important statements are either introduced or checked by experiment. In order to facilitate such treatment special experiments have been devised so that the total number of essential experiments may be kept as small as possible.

Except for the experienced practitioner and the advanced student, experimental work will be the main channel by which the student gains a practical and organised body of knowledge and experience of the nature of the various materials, and also of the departures which in practical construction are made from the ideal

conditions assumed in developing the subject-matter of this volume. Similarly it is important that, as with the growth of his industrial experience he is able gradually to reduce his dependence upon the conduct of laboratory experiments, the student should come to look upon full-sized constructional work as an extension of the simpler methods of the laboratory, and seek to analyse along similar lines the knowledge and experience so gained.

In the absence of a generally accepted standard notation the authors have adopted a scheme (see Appendix II) which is simple and reasonably in accordance with good practice at the present time.

The authors desire to express their thanks to the Governors of the Manchester Municipal College of Technology for facilities which have enabled them to carry out experimental work and for the use of the testing laboratories; their thanks are also due to the following firms for permission to use certain blocks and photographs: Messrs G. Cussons, Ltd., Manchester; Mr A. Macklow Smith, Westminster; and to the British Engineering Standards Association for permission to publish the tables printed in Appendix I.

J. L. M.
F. E. D.

July 1929

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SECTION I

THE CONDITIONS OF STATICAL EQUILIBRIUM, WITH APPLICATIONS TO BUILDING STRUCTURES

CHAPTER I

INTRODUCTION. THE CONDITIONS OF EQUILIBRIUM FOR SYSTEMS OF CONCURRENT COPLANAR FORCES

1. Introduction to Section I. Mechanics is one of the oldest of the sciences. Its origins are of peculiar interest to students of Building, since they are to be found in man's early attempts to devise contrivances for the raising and moving ot heavy weights, tasks which had to be faced, for example, when he undertook to erect the monumental buildings of early historical times. The term **Mechanics** (or Applied Mechanics) is now generally applied to the more practical aspects of the subject of **Dynamics**, the mathematical science which treats of the *motion* and *equilibrium* of material bodies under the action of forces.

The subject of **Mechanics** is usually divided into two main sections:

(1) **Kinetics** (sometimes also called Dynamics), which deals with systems of forces which produce *motion*; and

(2) **Statics**, which deals with systems of forces which produce *equilibrium, or a state of rest.*

In this volume we propose to investigate all the more important principles of Statics, since in the design and erection of buildings we are almost exclusively concerned with bodies which are at rest, under the action of systems of forces arising from the weights of structures and from the external loads which they have to carry.

The work will be divided into three sections, as follows:

Section I will treat almost exclusively of the equilibrium of structures and of parts of structures under the action of systems of external forces.

Section II will deal with the effects of these external forces upon the materials of which the structures are composed, giving particular attention to the important case of loaded beams. In this section are introduced the subjects of **Elasticity**, which deals with the equilibrium of strained bodies, and the **Strength of Materials**,

which treats of the power of structural materials to support the forces applied to them and to maintain the equilibrium of the structure as a whole.

Section III will carry the whole of the previous work a stage further by considering the application of statical knowledge to the main types of building construction.

2. Units for the measurement of statical forces. In the more general treatment given in Dynamics, **Force** is defined as *that which tends to change the state of rest or of uniform motion of a material body*, a definition which is derived from Newton's First Law of Motion. As explained in Vol. I, Chap. VI, however, it is possible to derive a satisfactory unit for the measurement of statical forces, which only involves the idea of motion by implication, from the definition of **Equal Forces**. *Two forces are said to be equal if, when applied to the same small body or particle but in exactly opposite directions, the particle remains at rest.*

Thus the force which will just support vertically the weight of one pound may be said to be "a force equal to one pound weight", or, more simply, "a force of one pound". Since the forces which occur most frequently in Statics are those due to weight, *we shall adopt the force of one pound as our statical unit of force.* (Any other weight or "gravitational" unit, such as the ton, the gram or the kilogram, may of course be used if more convenient.)

3. The Resolution of Forces. On the assumption that the reader is already acquainted with the elementary work in Statics which was included in Vol. I, it will be unnecessary to repeat that work in detail. As a convenient first step in our wider and more general consideration of the laws of statical equilibrium we may, however, reconsider the question of the resolution of forces, indicating at the same time the way in which the graphical methods described in Vol. I may be usefully supplemented by arithmetical and trigonometrical methods.

Definitions. In this chapter we will limit our treatment to systems of concurrent forces. Forces are **Concurrent** when their directions meet at a point, and **Non-concurrent** when they do not meet.

We have already seen (Vol. I, Chap. VI) that, where two (or more) forces are applied to the same particle—or have the same point of application—it is possible to obtain a single force, or **Resultant**, which would have the same effect upon the particle as the forces which it replaces.

Conversely, given a single force it is possible to "resolve" that force into two (or more) component forces, or **Components**, acting along any two (or more) lines drawn through the point of application of the single force.

If in the first case a force is applied to the particle which is equal in magnitude but acts in the opposite direction to the resultant, then this force is known as the **Equilibrant**. When acting with the original forces the equilibrant produces a state of rest or equilibrium of the particle.

I. The Parallelogram of Forces. If two forces having the same point of application be represented in magnitude and direction by the adjacent sides of a parallelogram drawn from (or to) their point of application, then their resultant shall be represented by the diagonal of the parallelogram drawn from (or to) that point.

When the lines along which the components of a given force act are at right angles to each other, then the components are known as the **Rectangular Components** of the force; see Fig. 1. This is an important case and of frequent occurrence; the following examples deal with rectangular components.

Example I. *A force of* 10 *lbs. acts at* O *along the line* OM *and away from* O, *as shown in Fig.* 1. *It is required to find its components along the lines* ON *and* OP, *which are at right angles to each other.*

(*a*) **Graphical solution.** Mark off OM to a suitable scale to represent 10 lbs. Draw MN parallel to OP and MP parallel to NO. Then by the converse of the Parallelogram of Forces the components of the force OM are the force ON, acting vertically along the line ON, and the force OP, acting horizontally along the line OP. The magnitude of each of these forces is obtained by measuring the lengths of the lines ON and OP respectively on the force scale used in marking off the force OM.

By actual measurement we get:

$$\text{Component force } OP = 8 \text{ lbs.}$$
$$\text{Component force } ON = 6 \text{ lbs.}$$

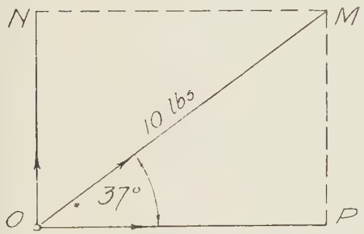


Fig. 1.

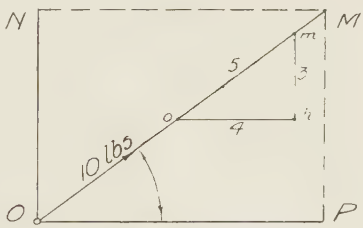


Fig. 2.

(*b*) **Trigonometrical solution.** See Fig. 1. Since the triangle MOP is a right-angled triangle, we have that

$$\begin{aligned} OP &= OM \times \text{cosine } 37° \\ &= 10 \times 0{\cdot}7986 \\ &= 7{\cdot}986 \text{ lbs.} \end{aligned}$$

Similarly

$$\begin{aligned} ON = PM &= OM \times \text{sine } 37° \\ &= 10 \times 0{\cdot}6018 \\ &= 6{\cdot}018 \text{ lbs.} \end{aligned}$$

(*c*) **Arithmetical solution.** See Fig. 2. As will be found at a later stage, this is a rapid and powerful method for dealing with framed structures having rectangular outlines, the relations between the forces acting in the various members being obtained from a consideration of their relative lengths. For the sake of simplicity the ratios of the sides of the

small triangle *mop*, which gives the inclination of the force OM, have, in the present example, been expressed in whole figures which give an inclination only approximately equal to 37°.

Since the figure $OPMN$ is a rectangle, $MP = ON$.

Also, since triangles MOP and mop are similar triangles, evidently

$$\frac{OP}{OM} = \frac{op}{om} = \frac{4}{5}.$$

Now OM represents 10 lbs., therefore

$$OP = \frac{4 \times OM}{5} = \frac{40}{5} = 8 \text{ lbs.}$$

Similarly $$ON = MP = \frac{mp}{om} \times OM = \frac{3 \times 10}{5} = 6 \text{ lbs.}$$

Example 2. *To find a force from its rectangular components. In Fig. 2 let* ON *and* OP *be the known rectangular components of the force* OM, *of which we know only the direction and require to know the magnitude.*

Since the figure $OPMN$ is a rectangle, $MP = ON$. Also ON is the vertical component of the force OM.

Since the triangle MOP is a right-angled triangle, we know that

$$OM^2 = OP^2 + PM^2$$
$$= OP^2 + ON^2;$$

hence we have that:

$$(\text{force } OM)^2 = (\text{component } OP)^2 + (\text{component } ON)^2,$$

or $$\text{Force OM} = \sqrt{\text{OP}^2 + \text{ON}^2}. \qquad \text{......(i)}$$

Example 3. *Rectangular component of a force along a line at right angles to its own length.*

Let the force OM in Fig. 3 act at O in the line OP, OM being at right angles to OP.

Using Fig. 1, the vertical component ON in this case coincides with and is equal to the force OM, hence the horizontal component along OP must be equal to zero, or, *the rectangular component of a force along a line at right angles to its own direction is zero.*

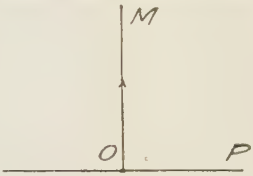


Fig. 3.

4. The rectangular components of two forces and their resultant. See Fig. 4 (A). Let F_1 and F_2 be any two forces and R be their resultant, obtained from the parallelogram $ONMP$, and let OX and OY be *any* two axes at right angles to each other.

Draw Nn, Pp and Mm at right angles to OX, and draw Pp' at right angles to OY. Then On, Op and Om are the rectangular components of F_1, F_2 and R along the axis OX.

Now since in the triangles ONn and PMp' the sides ON and PM are equal and the corresponding sides of each triangle are parallel, therefore the triangles are equal and $On = Pp' = pm$. Therefore

Om, the rectangular component of R, $= Op + pm = Op + On$, or
$=$ the sum of the rectangular components of F_1 and F_2. ...(a)

In the case illustrated in Fig. 4 (B), where OY falls within the angle made by the two forces F_1 and F_2, if components acting towards the left be called negative, we have, since $On = Pp' = pm$,

$$Om = Op - pm = Op - On; \qquad \text{......(b)}$$

or, the rectangular component of R is equal to the *algebraic* sum of the rectangular components of the forces F_1 and F_2.

Again, the above relation must hold even when there are more than two concurrent forces, since in such a case we may first find the resultant of one pair of forces and then add this resultant to

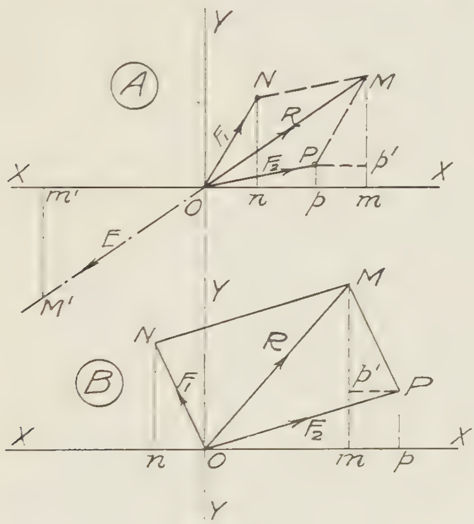


Fig. 4.

the next force, and so on until we are finally left with two forces and their resultant. Hence we may make the following general statement:

II. The algebraic sum of the rectangular components of two (or more) concurrent forces in any direction is equal to the rectangular component of their resultant in the same direction.

5. The rectangular components of a system of forces in equilibrium. In the case of a system of forces which is in equilibrium, since the resultant is zero the algebraic sum of the rectangular components in *any* direction must of necessity be also zero. We may, however, consider the simple case shown in Fig. 4 (A), when we replace R by the equilibrant E; then, since E is equal and opposite to R, Om', the rectangular component of E, must be

equal and opposite to Om, that is to $(Op + On)$, see (a) in para. 4. Hence in this case the algebraic sum of the rectangular components is zero. The same may be similarly shown to hold for larger systems. Hence:

III. If a system of concurrent forces be in equilibrium, the algebraic sum of the rectangular components of the forces in any direction is zero.

Example 1. See Fig 5. *Let the three forces* MO, NO *and* PO *acting at* O *be in equilibrium. If* MO *be a force of* 1000 *lbs. of the given inclination and the directions of* NO *and* PO *are as indicated, find the magnitude of the force* NO *and of the force* PO.

Produce PO and NO to X and Y respectively.

Mark off MO to represent force MO to some suitable scale.

Draw MR and MS parallel to OX and OY respectively.

By the methods previously adopted it is easy to show that RO, which is the vertical component of MO, equals 600 lbs.

Similarly SO, which is the horizontal component of MO, equals 800 lbs.

Then, for equilibrium along the vertical line NOY, force NO must be equal to force RO (600 lbs.) and act upwards.

Similarly, for equilibrium along the horizontal axis XOP, force PO must be equal to force SO (800 lbs.) and act towards the right.

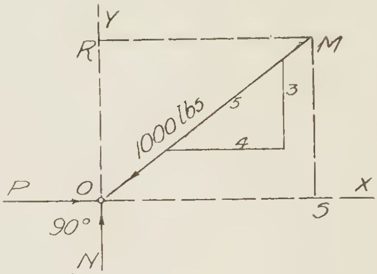


Fig. 5.

Example 2. See Fig. 6 (A). *Let the three forces* OM, NO *and* OP *acting at the point* O *be in equilibrium. The magnitude of the force* OM *being given, find the magnitudes of the other two forces.*

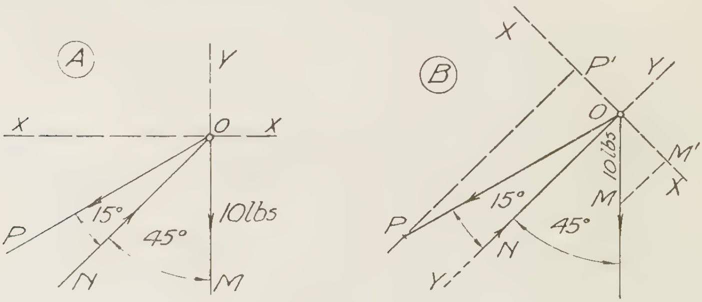


Fig. 6.

It will be evident from an inspection of the diagram that the vertical components of the inclined forces NO and OP acting along the line MOY must together balance the vertical force OM. but since it is impossible at

this stage to say what proportion of the total force would be contributed by NO and what by OP we must proceed differently. Instead of using vertical and horizontal axes we may use one of the inclined forces (NO) as the OY axis and draw the OX axis at right angles to it; see Fig. 6 (B).

If OM and OP represent these two forces to some scale and OM' and OP' are their components along the OX axis, then it is clear that, for equilibrium along this axis, these two components must be equal.

Mark off OM to some suitable scale to represent force OM.

Draw MM' parallel to YOY.

By construction angle MOM' is an angle of 45°, therefore

$$\begin{aligned} OM' &= OM \times \text{cosine } 45° = 10 \times 0{\cdot}707 \\ &= 7{\cdot}07 \text{ lbs.} \qquad \ldots\ldots\text{(a)} \end{aligned}$$

Hence OP' must also represent a force of 7·07 lbs.

By construction angle POP' is an angle of 75°, therefore

$$\begin{aligned} P'P &= OP' \times \text{tangent } 75° \\ &= 7{\cdot}07 \times 3{\cdot}732 \\ &= 26{\cdot}4 \text{ lbs.} \qquad \ldots\ldots\text{(b)} \end{aligned}$$

The magnitude of force OP can now be obtained from (a) and (b); since

$$(OP)^2 = (OP')^2 + (P'P)^2,$$

therefore

$$\begin{aligned} OP &= \sqrt{7{\cdot}07^2 + 26{\cdot}4^2} \\ &= 27{\cdot}3 \text{ lbs.} \end{aligned}$$

Consider next the components of the forces which act along the line NOY; there are evidently three such components, viz. NO acting towards O, a force equal to $P'P$ and representing the component of OP acting away from O, and also a force equal to $M'M$ and representing the component of OM acting in the same direction as $P'P$.

Since the force NO acts in the opposite direction to the components $P'P$ and $M'M$, NO will be equal in magnitude to the sum of $P'P$ and $M'M$.

Therefore $\quad NO = 26{\cdot}4 + 7{\cdot}07 = 33{\cdot}47$ lbs.

6. The general conditions of equilibrium for systems of concurrent forces. In the two examples which were worked out in the preceding paragraph we saw that, if the algebraic sum of the components of three forces in equilibrium were taken along two axes at right angles to each other, then the sum in each of these directions was zero. This is an important result and one of quite general application. Before summarising it, however, in a more precise form, we will consider the case from another aspect.

Case I. *A system of three concurrent forces in equilibrium.*

IV. The following statement of the **Triangle of Forces** is taken from Vol. I.

If three forces having the same point of application are in equilibrium, then any triangle whose sides are parallel to the directions of the forces will have the lengths of those sides proportional to the magnitudes of the forces.

The Converse of the Triangle of Forces is equally true.

In Fig. 7 let the three forces OM, ON and OP be in equilibrium. The forces being fully defined, let a force triangle abc be drawn in the usual way, the sense or direction of each force being indicated thereon.* Enclose the force triangle in *any* rectangle $a'bb'c'$ as shown.

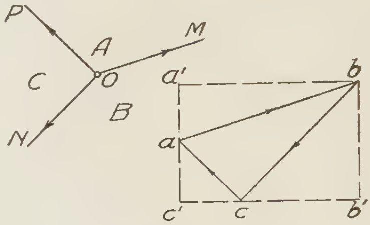


Fig. 7.

Then the horizontal component of force ab is represented by $a'b$ and acts towards the right. Similarly the horizontal components of forces bc and ca are $b'c$ and cc' respectively and act towards the left.

Since $a'bb'c'$ is a rectangle, the lengths $b'c$ and cc' are together equal to $a'b$; hence the algebraic sum of the "horizontal" components of the three forces is zero.

Similarly $c'a$ and aa', the components acting upwards, are together equal to bb', the component acting downwards; hence the algebraic sum of the "vertical" components is also zero.

Case II. *A system having more than three forces.*

V. The following statement of the **Polygon of Forces** is taken from Vol. I.

If any number of forces having the same point of application are in equilibrium, then a closed polygon may be drawn whose sides shall represent these forces in magnitude and direction.

The Converse of the Polygon of Forces, though important, cannot be expressed in the same general terms, since the polygon cannot be drawn if the magnitudes of more than two of the forces are unknown, but we can say that, *for equilibrium, the Force Polygon must be a "closed" figure.*

Let the five forces given in Fig. 8 be in equilibrium and let a corresponding force polygon $abcde$ be drawn to a suitable scale, the sense or direction of each of the forces being indicated as before.

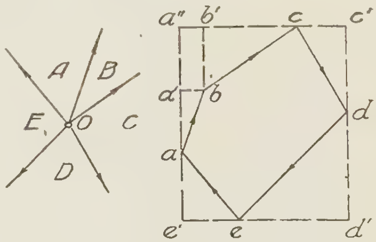


Fig. 8.

Enclose this figure in *any* rectangle $a''c'd'e'$ and draw the lines ba' and bb' parallel to the sides of the rectangle.

Then, as before, the "horizontal" components of the forces ab, bc and cd are $a''b'$, $b'c$ and cc' and act towards the right.

Similarly the "horizontal" components of the remaining forces de and ea are $d'e$ and ee' and act towards the left.

* For the lettering of the forces and force triangle see Bow's Notation in Vol. I, Chap. VI. See also para. 17.

Since the figure $a''c'd'e'$ is a rectangle, it is evident that $a''c'$ (that is, the sum of the "horizontal" components acting to the right) is equal to $d'e'$ (that is, the sum of the components acting to the left), from which it follows that the algebraic sum of the "horizontal" components must be zero.

In a similar manner, if we consider the "vertical" components acting upwards and those acting downwards, we may show that the algebraic sum of the "vertical" components is likewise zero.

The above proof is quite general, since it may be shown to be unaffected by the shape of the polygon or the direction of the sides of the enclosing rectangle.

Hence it would appear that what we may call the arithmetical or mathematical method of checking the equilibrium of a system of concurrent forces, by equating the rectangular components to zero, is *equivalent* to the graphical method, which requires that the force polygon must be "closed" to ensure that the system is in equilibrium.

In practice either method may be used which is most convenient, or one may be used as check upon the other.

The same results would of course be obtained if the two axes selected formed any other angle than a right angle, provided always that the rectangular components are taken in the two directions selected (see para. 5, III), and that the axes do not coincide. It is, however, usually more convenient to use two axes at right angles. In all cases we shall continue to refer to the "vertical" and "horizontal" components. No confusion need arise if these terms are taken to include the case of rectangular components in two directions not at right angles to each other.

VI. Conditions of equilibrium for systems of concurrent coplanar forces.

In order that a particle, acted on by a system of concurrent coplanar forces, may be in equilibrium it is necessary and sufficient that the sum of the components of these forces in each of two directions shall be separately zero.

Or, alternatively, the Force Polygon must close.

More briefly,

If V and H stand for the "vertical" and "horizontal" components of these forces respectively, and the sign Σ stands for "the algebraic sum of all such quantities as...",

then, for equilibrium:

The sum of the "vertical" components must be equal to zero, or $\Sigma V = 0$;

Similarly the sum of the "horizontal" components must also be equal to zero, or $\Sigma H = 0$.

The principles enunciated above may be applied in the solution of the following problems, the sum of the components in each of two directions being equated to zero.

Problems I

(**Note.** As each of the following problems contains a special difficulty which may be met with in practice, they should be worked in the order given.)

1. Fig. *A* represents the three forces acting at the foot of a roof truss. The magnitude and sense of the force *RO* is given and the sense of the force *SO* is known. Find the magnitude of the force *SO* and the magnitude and sense of the force acting along the line *TO*.

2. Fig. *B* gives the sense and direction of the forces acting at the head of a jib crane. The magnitude of the load being supported by the chain is 10 tons. Find the magnitudes of the forces acting in the jib and the tie.

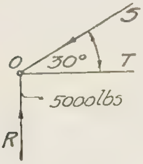


Fig. *A*.

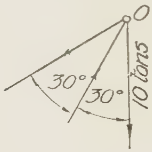


Fig. *B*.

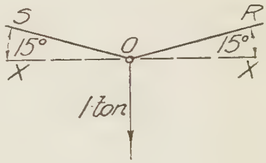


Fig. *C*.

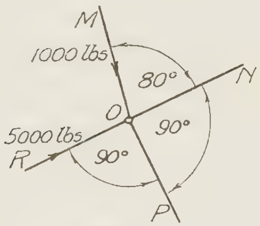


Fig. *D*.

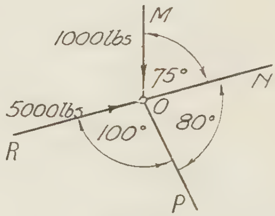


Fig. *E*.

3. Fig. *C* represents a load supported by two chains equally inclined to the horizontal. Find the forces acting in the two chains.

4. Fig. *D* represents the forces acting at a joint *O* in a roof truss. The sense and magnitude of each of two forces are given and the directions of the two remaining forces. Find the magnitude and sense of the forces in *NO* and *PO*.

5. Fig. *E* represents a similar case to that shown in Fig. *D* with the exception that the member *OP* is not at right angles to the line *RON*. Find the magnitude and sense of the force in *NO* and in *PO*.

CHAPTER II

SYSTEMS OF NON-CONCURRENT FORCES. CONDITIONS OF EQUILIBRIUM FOR ALL SYSTEMS OF COPLANAR FORCES

7. Non-concurrent forces. In the consideration of the equilibrium of systems of concurrent forces, since the *position* of each of the forces is determined by the fact that each has the same point of application, it is only necessary to deal with the remaining qualities of each force, viz. its *magnitude* and its *direction*. In such cases if the forces are not in equilibrium then the particle to which they are applied will tend to move in one direction or another, i.e. the movement would be one of **translation**.

When we come to consider the application of systems of forces to extended bodies, such as levers, beams and trusses of various shapes, in which the forces do not all act at the same point, i.e. they are *non-concurrent forces*, then it is apparent that, in addition to the *magnitude* and the *direction* of each of the applied forces, we must also consider the effect of the *position* of each of the forces.

8. The Principle of Moments. In the preliminary consideration of this type of problem in Vol. I, it was shown that this new condition—that of the position of a *non-concurrent* force—led to the introduction of a new magnitude, termed the *moment of a force*.

Definition. The moment of a force F about any point P, see Fig. 9, is equal to $F \times d$, where d is the perpendicular distance of P from the line of action of the force F. The moment of a force may be *clockwise* or *anti-clockwise*, according as it tends to turn the body to which it is applied in one direction or the other. The moment of a force will be measured in units of "pound-inches" (lb. ins.) and "ton-feet" (ton ft.), according to the units used for weight and length.

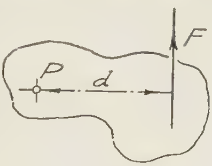


Fig. 9.

It was further shown that, for a body to be in equilibrium under the action of a number of non-concurrent coplanar forces, the moments of these forces must be balanced among themselves, otherwise there would be a movement of **rotation** which would be inconsistent with the condition of equilibrium. All this is contained in the statement of the Principle of Moments which, because of its importance, may be re-stated here.

I. The Principle of Moments. If a body be in equilibrium under the action of a number of coplanar forces, then the algebraic sum of

the moments of the forces about any point in the plane of the forces will be zero.

Or, more briefly,

If M stands for the moment of each of the forces, then $\Sigma M = 0$.

9. Experiment. *To find the conditions of equilibrium for a body under the action of a number of non-concurrent forces* (a) *when the forces are inclined to each other, and* (b) *when the forces are parallel forces.*

(*a*) The equilibrium of inclined non-concurrent forces. Take a stout card or thin board of irregular shape and, by means of cords, pulleys and weights, arrange to support it in a vertical position as shown in Fig. 10. If the body be not light the effect of its weight should be included, otherwise the accuracy of the experiment will be affected. One method of overcoming this difficulty is to find the centre of gravity (C.G.) of the body and take its weight as acting vertically at that point. In practice it is better to attach a weight at C.G. and to take the total weight (W) at that point to be equal to the weight of the body (which is not usually considerable) plus that of the attached weight; the inclination of the other forces can then be conveniently referred to this vertical line. An alternative plan is to support the body horizontally on steel balls as described in Vol. I, Chap. VII, when the weight of the body need not be considered.

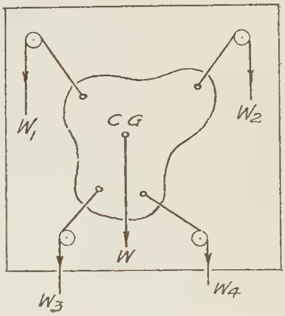


Fig. 10.

By means of a sheet of paper placed behind the body and cords, obtain the direction and position of each of the forces and note the magnitudes as given by the weights.

Remove the paper and carry out the following investigations:

(i) Marking on the paper a vertical and a horizontal axis (or any other two convenient axes at right angles to each other), find whether the sum of the "vertical" components and the sum of the "horizontal" components are respectively zero, that is, find whether

$$\Sigma V = 0, \text{ and } \Sigma H = 0.$$

Or, alternatively, ascertain whether a closed force polygon can be drawn for the system of forces (see para. 6). If either of these conditions are satisfied, we know that there can be no movement of *translation*.

(ii) Select *any* point on the paper and measure the *perpendicular distance* between that point and each of the forces acting on the body. Tabulate the results as follows and obtain the value of the moment for each of the forces, noting whether the moments are clockwise or anti-clockwise.

Force (lbs.)	Distance (ins.)	Force × distance (lb. ins.)	
		Clockwise	Anti-clockwise
	Totals		

The algebraic sum of the values in the last two columns should show that $\Sigma M = 0$, thus indicating that there can be no *rotational* movement.

(*b*) **The equilibrium of parallel forces.** Let a beam be supported and loaded as shown in Fig. 11 so that the supporting forces S and T can be measured. Tabulate the forces as before and note their positions. (Allow for the weight of the beam by taking the initial readings of the balances as "zero" when the beam is unloaded.)

(i) Find whether $\Sigma V = 0$ and $\Sigma H = 0$.

As suggested in (*a*) a force polygon may be drawn in order to test whether the vertical and horizontal components are in equilibrium. When this polygon is completed, the lines representing the forces will be found to lie on a vertical line. This obviously must be so, since all the forces are vertical forces; for the same reason there cannot be any horizontal components and ΣH must therefore be equal to zero.

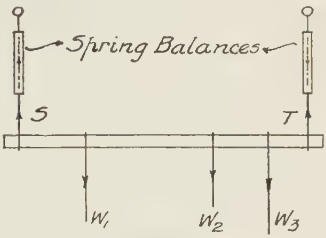


Fig. 11.

(ii) Taking any point in the beam, or outside of it, measure and tabulate the forces and their distances from the selected point.

The calculated values for the moments should be tabulated as before and the result should show that the algebraic sum of the moments is zero, or $\Sigma M = 0$.

10. General statement of the conditions of equilibrium for any system of coplanar forces. From the experiments just completed we find that, in dealing with systems of non-concurrent forces, the additional factor to be considered is due to the moments of the forces. It is thus seen that in order to ensure the equilibrium of any system of coplanar forces—concurrent as well as non-concurrent—it is necessary that there should be no tendency either to translational or to rotational movement. (The second condition is of course always satisfied in the case of systems of concurrent forces which are in equilibrium, since the lines of action of the forces pass through one and the same point and ΣM must of necessity be zero.) We may thus put our statement of the necessary conditions of equilibrium into quite general terms as follows:

II. In order that a body acted upon by any system of coplanar forces may be in equilibrium it is necessary and sufficient that the sum of the rectangular components of these forces in each of two directions shall be separately zero, and also that the algebraic sum of the moments of the forces about any point in the plane of the forces shall be likewise zero.

Or, more briefly,

For equilibrium:

The sum of the "vertical" components must be equal to zero, or $\Sigma V = 0$;

Similarly the sum of the "horizontal" components must be equal to zero, or $\Sigma H = 0$;

And likewise the sum of the moments of all the forces about any point in the plane must be equal to zero, or $\Sigma M = 0$.

Systems of coplanar forces may be classified thus:

I. Systems of concurrent forces.

II. Systems of non-concurrent forces, which may be further subdivided into

(*a*) Systems of parallel forces;

(*b*) Systems of non-parallel or inclined forces.

The solution of problems. When dealing with problems concerning systems of coplanar forces in equilibrium, three separate equations can be set down, based upon the three conditions stated above, from which *not more than three unknown quantities* can be found if all the other details of the system are given. In the following worked example, which revises the work of this and the preceding chapter, these three equations have been indicated; it will be seen that they are fundamental to the solution of the problem.

Example. *In Fig. 12 is given the outline of the two masts and supporting cable of a simple type of concrete-placing plant. The cable is secured at each end to a mast—held in position by inclined stays—and from the cable hangs a series of jointed chutes or troughs (not shown). The concrete is poured into the upper end of the troughs and finds its way down by gravity to a placing-nozzle at the lower end.*

From the dimensions and weights indicated, determine:

(*a*) *The vertical thrusts* (V_A *and* V_B) *on the masts at* A *and* B*;*

(*b*) *The force* (F_A) *in the cable at* A*, the inclination being given;*

(*c*) *The horizontal components* (H_A *and* H_B) *of the pulls in the cables at* A *and* B *respectively;*

(*d*) *The magnitude and inclination of the force* F_B.

(*a*) Considering only the vertical forces and components, it will be seen from Fig. 12 (B) that we have a system of parallel forces in equilibrium. Taking moments about A we have, from $\Sigma M = 0$ (Equation 1),

$$V_B \times 120 = 800 \times 60 + 400 \times 75 + 400 \times 90 + 400 \times 105,$$

from which $$V_B = 1300 \text{ lbs.}$$

From $\Sigma V = 0$ (Equation 2), we have

$$V_A = (800 + 400 + 400 + 400) - 1300 = 700 \text{ lbs.}$$

(*b*) By the methods of Chap. I, using the small force triangle at A,

$$\frac{V_A}{F_A} = \sin 30°,$$

or $$\text{Total force at } A = F_A = \frac{V_A}{\sin 30°} = \frac{700}{0{\cdot}5} = 1400 \text{ lbs.}$$

(*c*) Since H_A and H_B are clearly the resultants of all the intermediate horizontal components of the forces in the cable and there are no other horizontal forces, also $\Sigma H = 0$ (Equation 3), it follows that H_A and H_B must be equal in magnitude. Then, by the methods of Chap. I,

$$H_A = F_A \times \cos 30^\circ$$
$$= 1400 \times 0{\cdot}866 = 1212 \text{ lbs.} = H_B.$$

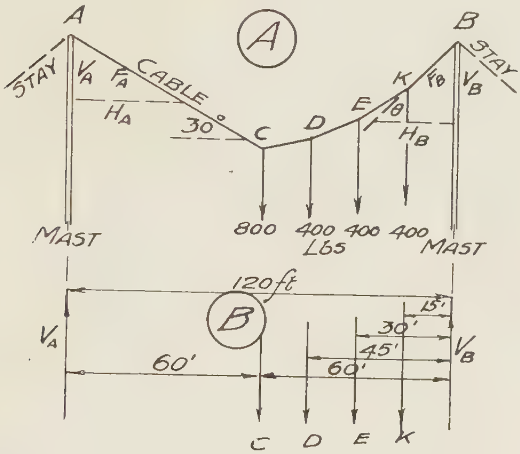


Fig. 12. Supporting forces in a concrete-placing plant.

(*d*) Since we have found both V_B and H_B, then from expression (i), para. 3, we have

$$F_B = \sqrt{V_B^2 + H_B^2}$$
$$= \sqrt{1300^2 + 1212^2} = 1780 \text{ lbs. (approx.).}$$

The inclination of F_B can be obtained from the shape of the force triangle at B, evidently

$$\tan\theta = \frac{V_B}{H_B} = \frac{1300}{1212} = 1{\cdot}072,$$

whence $\theta = 47^\circ$.

Problems II

1. For the loads and distances shown in Fig. *A*, Problems IV, find the supporting forces acting at *A* and *B* to produce equilibrium.

2. Find the supporting forces at *A* and *B* in the roof truss shown in Fig. *B*, Problems IV (No. 2).

3. In Fig. *A* is shown a ladder carrying a ladder-scaffold in the position shown. If it be assumed that the reaction at *A* is at right angles to the wall surface, find its value. What are the vertical and horizontal components of the reaction at *B*?

4. Fig. *B* shows a simple type of wall-scaffold, supported from a holder passed into a joint at *A*. If the reaction at *C* be assumed to be

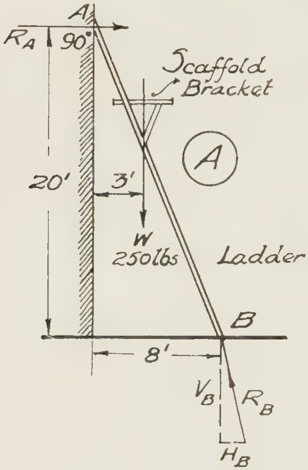


Fig. *A*. Ladder-scaffold.

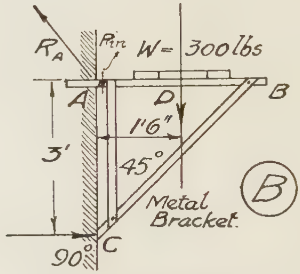


Fig. *B*. Wall-scaffold.

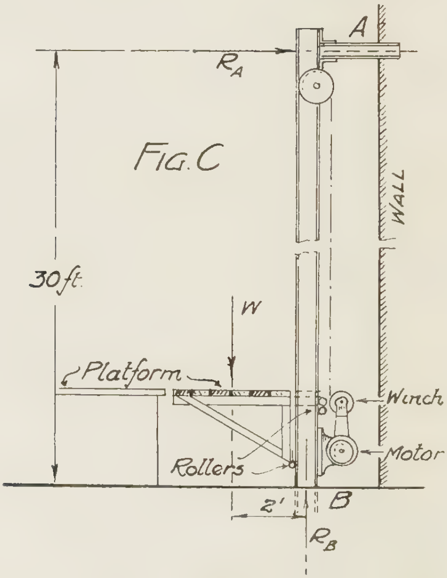


Fig. *C*. Lift for material.

at right angles to the surface of the wall, find its magnitude. Find also the vertical and horizontal components of the supporting force at *A*.

5. In Fig. 94 is shown the elevation of a derrick crane, when the jib and tie lie in the same plane as the side frame PBC. If the load W, of 5 tons, acts at 50 ft. horizontally from C, find the magnitude of the vertical supporting forces at B and C, if BC is 30 ft.

6. A lift for hoisting material is to be erected against the wall of an existing building (or alternatively against a scaffolding), as shown in Fig. C. The lift consists of a vertical frame made from two I-section steel girders. This rests on a foundation at B and is held from the wall at the top by the horizontal piece at A. The motor and winch are secured to the back of the upright girder as shown, the rope passing over a pulley at A and so down to the platform. The platform is carried by rollers resting on the vertical girder. From the dimensions given, find the horizontal force (R_A) at A, and the vertical and horizontal components of the supporting force (R_B) at B, when the load W on the platform is 2000 lbs. What is the inclination of the force R_B?

CHAPTER III

PROBLEMS IN PARALLEL FORCES. CENTRES OF GRAVITY AND CENTROIDS

11. To find the resultant of two like parallel forces.

Experiment. Suspend the beam AB by two spring balances as shown in Fig. 13. Take the initial readings of the balances as zero.

From A and B suspend two weights S and T. Note the increases in the readings of the spring balances; obviously they will be equal to the magnitudes of S and T respectively.

Since S and T act vertically downwards and have no horizontal components, the magnitude of the resultant of these two forces must evidently act vertically and be equal to their sum. Remove the weights S and T and suspend from the beam a single weight R, equal in magnitude to $(S+T)$. Move the new weight along the beam until, if possible, a point C is found such that, when R is suspended at that point, the readings of the spring balances are the same as when the separate weights S and T were suspended at A and B respectively.

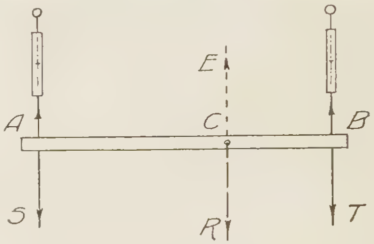


Fig. 13.

Measure the distances AB, AC and CB, and tabulate these together with the magnitudes of the weights and supporting forces as shown below.

Repeat the experiment several times with new weights and distances, recording the results thus:

Distance AC	Distance CB	Force S	Force T	$\frac{AC}{CB}$	$\frac{T}{S}$	$S \times AC$	$T \times BC$

The results should show (a) that the force R acting at C is nearer the larger force; (b) that C divides AB into two parts which are inversely proportional to the weights S and T; and (c) that the two forces have equal moments about C.

That the last two statements are alternate ways of stating the same fact may be readily shown as follows:

Assume that R is replaced by an equal and opposite force E; see Fig. 13. The three forces S, T and E would be in equilibrium and $\Sigma M = 0$. Take moments about C. Since the moment of E about C is zero, we have $S \times AC = T \times CB$, or

$$\frac{S}{T} = \frac{CB}{AC}.$$

Graphical construction to find *C*. Taking values obtained in the experiment just completed, set out the positions to scale as shown in Fig. 14. Draw at any convenient position a line ab (which need not be horizontal). Set *up* on the line of action of force S a distance aa' which represents to scale the *opposite force* T. Similarly set *down* from b a distance bb' to represent to scale the *opposite force* S.

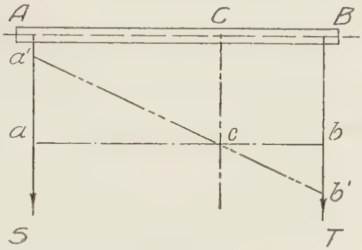


Fig. 14.

Join $a'b'$ cutting ab in c. Then c is on the line of action of the resultant R and the point C may be obtained by drawing cC parallel to Bb to cut AB in C.

Check the position of C found in this way by the measurements obtained experimentally.

It is easy to show geometrically that the construction we have used above will divide AB at C in such a way that the lengths AC and CB are in inverse proportion to the weights acting respectively at each end of the line.

Since aca' and bcb' are similar triangles, we have that

$$\frac{ac}{aa'}=\frac{bc}{bb'}, \quad \text{or,} \quad \frac{ac}{bc}=\frac{aa'}{bb'}=\frac{T}{S}.$$

12. To find the resultant of two unlike parallel forces.

Experiment. Arrange the support of a long beam in the manner shown in Fig. 15 (A). The length of the beam should be so disposed that, without any other load, both balances show a reading. These two readings should be taken as the "zero" readings for the supporting forces.

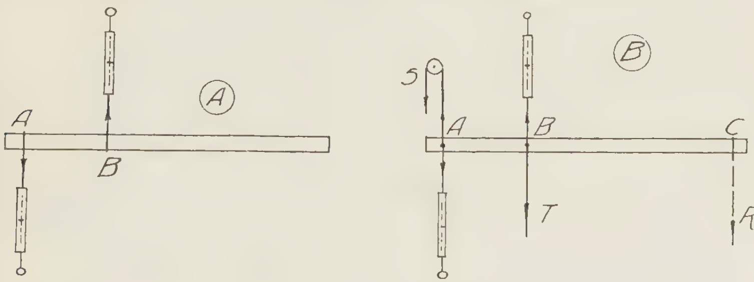


Fig. 15.

Add forces S and T acting in opposite directions at A and B respectively as shown in Fig. 15 (B). The downward acting force should be the greater of the two. Take the readings of the two balances. There are no horizontal components of the forces acting on the beam and it should be fairly obvious that the resultant R will therefore be equal to $(T - S)$ and act in the direction of the larger force.

Remove forces S and T and endeavour to apply a force equal to R at some point C so that the readings of the two balances are as before. When such a point has been found, measure the distances AC and CB and tabulate the results as explained in the last experiment. Repeat the experiment with other weights and distances.

The results should show: (*a*) that R acts outside the larger force; (*b*) that R is equal in magnitude to the difference between the two forces; (*c*) that C is so placed that the distances to A and B are inversely proportional to the weights acting at those points; and (*d*) that the forces S and T have equal moments about the point C.

Graphical construction to find C. Set down the lines of action of S and T as before; see Fig. 16. Draw the line ab anywhere (ab need not be horizontal) and *set up* the distance aa' to represent the opposite force T, *also set up* the distance bb' to represent the opposite force S. (Compare this procedure with that previously adopted when dealing with two like forces.)

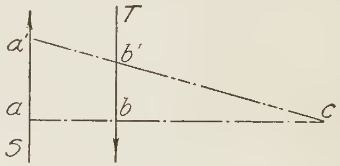


Fig. 16.

Join a' and b' and produce to meet ab produced in c. Then c is a point on the line of action of the resultant R. Check the result obtained in this way against that obtained experimentally.

The geometrical proof that ac and cb are related to each other in the way we require is similar to that already given for like forces and is left as an exercise for the reader; see Problems III, 1.

Note. This construction is very inconvenient when S and T are nearly equal in magnitude; it fails altogether when the forces are equal, for reasons which will be discussed later.

13. To find the resultant of three or more like parallel forces.

Before proceeding to the discussion of a more powerful graphical method for the solution of this type of problem, it will be useful to apply the simple methods already dealt with to the solution of a number of practical problems.

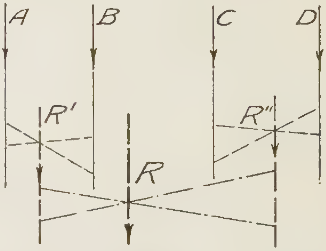


Fig. 17.

(*a*) **Graphical method.** To obtain the resultant of a number of parallel forces by this method it is convenient to deal with the forces in pairs, to obtain their resultants, and to combine these resultants until finally a single resultant is obtained. The diagram in Fig. 17 will make this clear.

Four parallel forces A, B, C and D are given; it is required to find their resultant.

Forces A and B are first combined and give the resultant R'.

Similarly forces C and D give the resultant R''.

R' and R'' are then combined to give the final resultant R.

Provided that the work is set out clearly, no difficulty need be experienced in applying this method, but it is sometimes an advantage to employ an enlarged horizontal scale.

The treatment of unlike forces is left as an exercise for the reader.

(*b*) **Arithmetical method.** A little consideration will show that, if a system of parallel forces is to be replaced by one force, viz. the resultant, then, if the "turning effect" is to be the same, the moment of the resultant must be equal to the algebraic sum of the moments of all the other forces, which the resultant replaces, about any point which may be selected in the plane of the forces. Thus in Fig. 18, in symbols, if Y be the perpendicular distance of the resultant (R) from some convenient point P, while y stands for each similar distance measured from P to each of the other forces (F), then $R \times Y = \Sigma F \times y$. From which we have

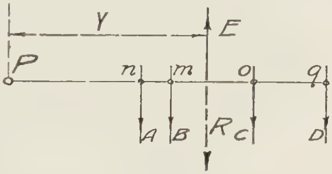


Fig. 18.

$$Y = \frac{\Sigma F y}{R},$$

or, in words,

The distance from P to the resultant

$$= \frac{\textbf{sum of the moments of all forces about } P}{\textbf{magnitude of the resultant}} \quad \ldots\ldots\text{(i)}$$

This expression can be applied to all systems of parallel forces—like and unlike—and enables us to find a point in the line of action of the resultant. The point P can often be conveniently taken in the line of action of one of the forces.

14. Centres of Gravity, Centres of Area, and Centroids. In some statical problems it is necessary to find the **Centre of Area** or **Centroid** of the cross section of a member, such as a beam or wall. If we think of the section as a thin slab of uniform thickness and density, then the centre of area or centroid corresponds in position to the centre of gravity of the slab. The position of this point may be found either by the use of the method of suspension described in Vol. I, Chap. VII, by drawing, by calculation,

or by a combination of the last two methods. The following facts were noted in Vol. I and may be repeated here:

(*a*) If any figure has an axis of symmetry, then the centre of area or centroid will lie on that axis;

(*b*) If it possesses two axes of symmetry, then the centroid will lie at the intersection of these two axes;

(*c*) The centroid of a triangular figure lies at the intersection of two of the medians of the triangle; see point G, Fig. 19 (A). (A median is a line drawn from one corner of a triangle to the centre of the opposite side.)

It may be shown that the point G divides the median into two parts such that CG is twice the length of Gc, or in other words Gc is one-third the length of Cc.

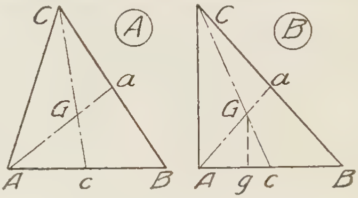


Fig. 19.

The case of the right-angled triangle is important. In Fig. 19 (B) draw Gg perpendicular to the side AB.

Then since triangles ACc and gGc are similar, evidently

$$\frac{cg}{cA} = \frac{cG}{cC} = \frac{1}{3},$$

but $$cA = cB = \tfrac{1}{2}AB;$$

therefore $$cg = \tfrac{1}{6}AB, \quad \text{and } Ag \text{ must} = \tfrac{1}{3}AB,$$

which gives the position of G as measured horizontally from A.

15. The first moment of an area. Definition. *The first moment of an area about any axis is obtained by multiplying the area of the figure by the perpendicular distance from the selected axis to the centroid of the figure.* Or, if A be the area of the figure and Y the distance from the centroid to the selected axis, then

$$\textbf{First moment of the area} = A \times Y. \qquad \text{......(ii)}$$

Thus in the case of the rectangle shown in Fig. 20 (A), we may think of it rotating about an axis S–S along one edge, and, since the centroid G is $d/2$ from S–S and the area of the figure is $(b \times d)$, the first moment of the rectangle about S–S is $(d/2 \times bd)$, or $bd^2/2$.

Similarly if we have a figure, such as that shown in Fig. 20 (B), which is made up of a number of separate figures or shapes, then *the total first moment of the whole figure about* S–S *will be equal to the sum of the first moments of each portion.* Or, if a stands for the area of each portion, and y for the distance from S–S to the centroid of each portion, then

$$\textbf{The total first moment} = \Sigma\,(a \times y). \qquad \text{......(iii)}$$

If in Fig. 20 (B) the axis X X passes through the centroid G for the whole figure, and Y is the distance between S–S and

X–X, then, combining expressions (ii) and (iii) we have: First moment of whole figure $= A \times Y = \Sigma\,(a \times y)$, or

$$Y = \frac{\Sigma\,(a \times y)}{A},$$

or, in words,

The distance to the centroid from S–S

$$= \frac{\textbf{sum of the first moments of the separate figures about } S\text{–}S}{\textbf{area of the whole figure}} \quad \ldots\text{(iv)}$$

As the Example given below will show, we can use this expression to find the position of the centroid.

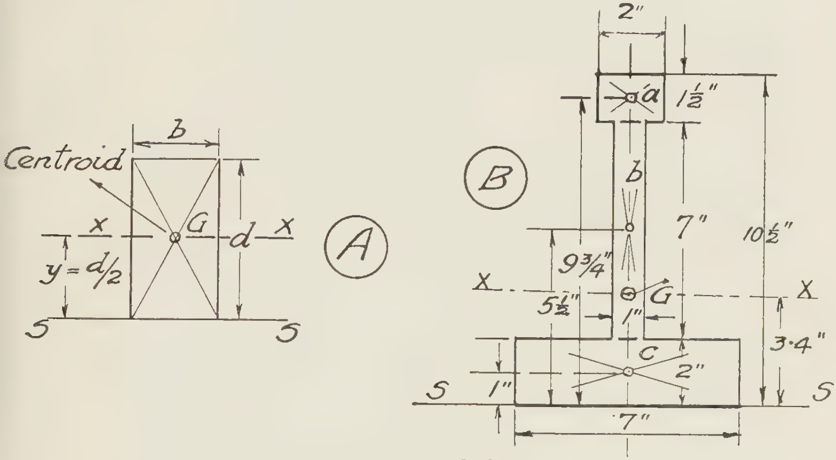


Fig. 20. The centroid of an area.

If we wish to find the first moment of any figure about an axis X–X which passes through the centroid, then, since in this case the distance Y becomes zero, we have, from (ii),

The first moment of an area about an axis through its centroid is zero. ......(v)

The truth of this statement may be demonstrated experimentally by balancing a cardboard figure of any shape upon a knife edge passing through the point corresponding to the centroid. Statement (v) is most important, but we shall not make use of it until we reach Chap. XIII.

Example. *To find the position of the centroid of the figure shown in Fig.* 20 (B), *which represents the section of a cast iron beam.*

Taking moments of each portion of the section about S–S, we have for *Area* (a): Area $= 2 \times 1\frac{1}{2} = 3$ sq. ins. Distance of centroid from S–$S = 9\frac{3}{4}$ ins. First moment about S–$S = 3 \times 9\frac{3}{4} = 29\frac{1}{4}$ inch units.

Area (*b*): Area = $7 \times 1 = 7$ sq. ins. Distance to centroid = $5\frac{1}{2}$ ins. First moment = $7 \times 5\frac{1}{2} = 38\frac{1}{2}$ inch units.

Area (*c*): Area = $7 \times 2 = 14$ sq. ins. Distance to centroid = 1 in. First moment = $14 \times 1 = 14$ inch units.

Total first moment of figure about S–S = $29\frac{1}{4} + 38\frac{1}{2} + 14 = 81\frac{3}{4}$ inch units.

Total area of figure = $3 + 7 + 14 = 24$ sq. ins.

Then if Y be the distance from S–S to axis X–X through the centroid G, we have

$$Y = \frac{\text{first moment of whole figure about } S\text{–}S}{\text{total area of figure}} = \frac{81\frac{3}{4}}{24} = 3{\cdot}4 \text{ ins.}$$

Since the figure is symmetrical about the vertical axis, the centroid will lie on this axis. (If the figure had not been symmetrical about a vertical axis, a second operation of the same kind would have been required to find the position of a vertical axis passing through the centroid. The intersection of the two axes would decide the position of the centroid.)

Problems III

1. Using the diagram given in Fig. 16, show that, if aa' represents force T to any scale and bb' represents force S to the same scale, then (*a*) $ac/bc = T/S$, and (*b*) $S \times ac = T \times bc$.

2. Fig. *A* represents a buttress which is 3 ft. thick throughout. The material of which it is constructed weighs 140 lbs. per cu. ft. Find the total weight of the buttress and the position of the vertical line in which the total weight may be considered to be acting. (In other words, find the vertical line which passes through the C.G. of the buttress.)

(**Note.** If vertical lines are drawn from *A* and *B*, the buttress will be divided up into three blocks the C.G. of each of which can be readily found. The problem then resolves itself into finding the resultant of three

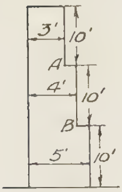


Fig. *A*.

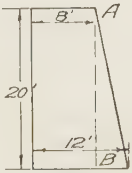


Fig. *B*.

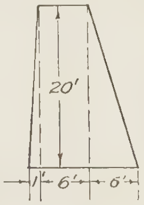


Fig. *C*.

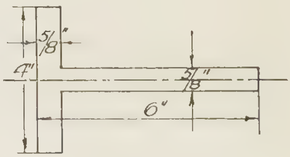


Fig. *D*.

vertical forces representing the weights of these blocks. Another method is to draw horizontal lines to *A* and *B* which again divide the buttress into three parts. The former method is, however, to be preferred, as in that case the lines of action of the vertical forces are farther apart, and the geometrical work can be more readily carried out. The problem should be solved both geometrically and arithmetically.)

3. Fig. *B* represents the section of a retaining wall. The weight of the material is 135 lbs. per cu. ft. Assume that a 1 ft. length of the wall is taken. Find the C.G. of this portion.

(**Note.** Draw the vertical line *AB*, and then find the positions of the C.G. of the rectangle and of the triangle into which the section is thus

divided. Then find the position of the resultant of the two forces due to the weights.)

4. Fig. *C* represents a retaining wall in which both faces are inclined to the vertical. The weight of the material is 135 lbs. per cu. ft. Find the vertical line in which the C.G. is situated.

(Both the graphical and arithmetical methods should be used.)

5. Fig. *D* represents the section of a steel tee bar (the rounded corners being neglected). Find the centroid of the section.

(**Note.** The figure divides easily into two rectangles. The dotted line is an axis of symmetry of the figure and it is only necessary to find the line at right angles to this line in which the centroid is situated.)

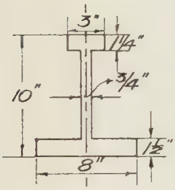


Fig. *E*.

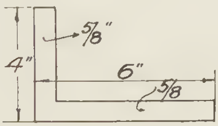


Fig. *F*.

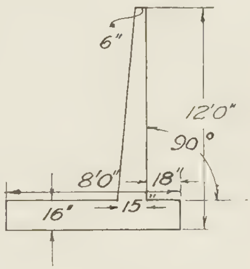


Fig. *G*.

6. Fig. *E* represents the section of a cast iron beam. Find the centre of area of the section.

7. Fig. *F* represents the section of an unequal-sided angle bar. Find the centroid.

(**Note.** Since this section does not possess an axis of symmetry, it will be necessary to find two lines at right angles to each other which both pass through the required point. The intersection of the two lines will obviously give the centroid, which in this case is outside the figure.)

8. Fig. *G* gives the shape of a certain type of retaining wall. Find the vertical line in which the centre of area is situated.

CHAPTER IV

THE GRAPHICAL TREATMENT OF SYSTEMS OF NON-CONCURRENT FORCES. THE LINK POLYGON AND POLAR DIAGRAM. REACTIONS OF TRUSSES

16. The mathematical and graphical treatment of systems of non-concurrent forces. In dealing with systems of concurrent forces in Chap. I we found that the graphical check upon equilibrium, which was given by the "closing" of the force polygon, was equivalent to the mathematical check obtained from the summation of the rectangular components of the forces in two directions; see para. 6. Having developed a similar mathematical check for systems of non-concurrent forces in Chap. II, it now remains for us to lay down a corresponding graphical method for dealing with such systems. This method is based upon the use of the **Link Polygon.**

As the succeeding chapters will indicate, the link polygon forms the basis of some very important sections of the graphical treatment of statical problems; it has therefore been thought worth while to devote a whole chapter to this method and in it to emphasise particularly the experimental aspect of the link polygon which, in spite of its proved usefulness in elucidating the meaning of the many applications of the link polygon, does not appear to have received the same attention as have the mathematical and geometrical aspects.

17. To find the resultant of a number of parallel forces by means of a link polygon. Four like parallel forces are shown in Fig. 21 (I), of which we require to find the magnitude and position of the resultant. Set out the positions of the forces on a "space diagram". Letter the spaces between the forces in accordance with Bow's Notation (see Vol. I, Chap. VI), placing and reading the letters in a clockwise direction around the lower part of the diagram; see Fig. 21 (I). Each force is then named by the letters on each side of the force, reading these letters in clockwise order.

Draw *any* line 1–2 between the forces BC and CD. Letter the space above 1–2 as P. Let us now assume that the line 1–2 represents a **Link** of any suitable material, in which there are two equal and opposite forces acting inwards at each end as shown by the arrow heads. Since these two forces are equal and opposite and act in the same straight line, their resultant must be zero; hence their introduction cannot affect the net result of the original pair of forces BC and CD.

Now consider the forces acting at point 2. These are force BC acting downwards, and force CP acting inwards, in the link. Set down on a separate diagram, see Fig. 21 (II), the vertical line bc to represent the force BC to some convenient scale. Next draw cp parallel to the link 2–1 and mark off the distance cp to represent the force CP assumed to be acting at the right-hand end of the link. (In practice the length cp should be selected so that angles between the lines radiating from p are all as large as practicable.) Join the points p and b, thus completing the force triangle pbc. Then evidently the line pb will represent the equilibrant of the forces represented by bc and cp and if, calling this new force E_2, we apply it at point 2, as shown in Fig. 21 (III), then the three forces now shown to be acting at 2 evidently form a system in equilibrium.

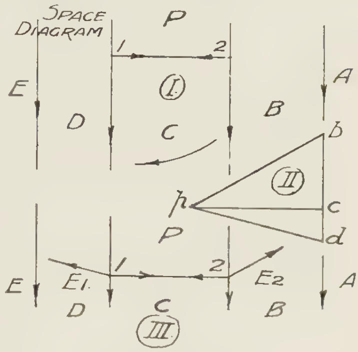


Fig. 21.

Similarly since pc (note the order in which this is read "p to c") represents the force PC acting at 1, which is at the other end of the link 1–2, we may set down cd on the diagram, Fig. 21 (II), to represent the force CD and join dp. Then evidently, as before, the line dp will represent the equilibrant of the forces represented by pc and cd and, if a force E_1, equal to this force, be applied at 1, as shown in the diagram, Fig. 21 (III), then the three forces acting at 1 will be a system in equilibrium.

Then evidently the two forces E_1 and E_2 are together sufficient to balance the two vertical forces BC and CD together with the two forces acting in the link 1–2 and which are represented by pc and cp.

There are still two forces AB and DE to be considered. For the sake of clearness an additional diagram has been given. On the new diagram, Fig. 22 (I), the lines of action of the forces E_1 and E_2 have been produced to meet the lines of action of the two outside forces DE and AB respectively in the points 3 and 4.

The lines 3–1 and 2–4 may be taken to represent two new links, in which the additional forces are those necessary to balance the forces already acting along these lines—that is E_1 and E_2 in Fig. 21 (III). In the diagram, Fig. 22 (II), these new forces are represented by pd and bp respectively.

Produce the line dcb and mark off ab to represent the force AB, and de to represent the force DE. Now consider the forces acting at 4. The force AB is represented by ab, while bp represents the

force acting in the link 4–2 away from 4. Then, joining p and a, the line pa evidently represents the equilibrant of these two forces. The line E_4 has been drawn to represent this force. The three forces acting at 4 are evidently in equilibrium.

Similarly ep will represent the equilibrant of the two forces acting at 3, since these forces are represented by the lines pd and de in the diagram, Fig. 22 (II), and we can draw the line E_3 to represent this force in the diagram, Fig. 22 (I).

We have shown that the added link forces do not affect the net result of the original forces; hence, ignoring the links 3–1, 1–2 and 2–4, we arrive at the condition shown in Fig. 22 (III), in which the system of four parallel forces together with the two inclined forces E_3 and E_4 form a system of forces in equilibrium.

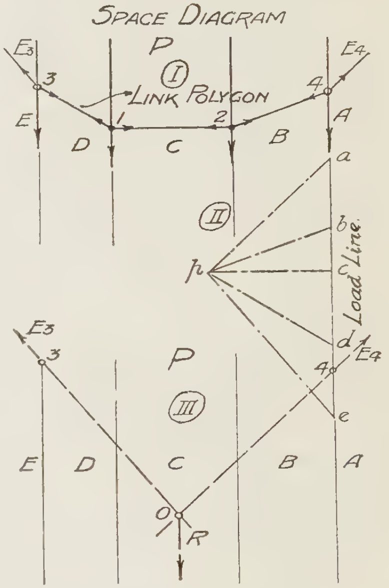


Fig. 22.

To find the position of the resultant. If the lines of action of the two inclined forces be produced to meet in some point O, then, if we apply at this point some force R of the correct magnitude and direction, it would be possible to form a system of three forces in equilibrium, R being the equilibrant of E_3 and E_4. But the four parallel forces are, as we have seen, capable of producing, together with E_3 and E_4, a system in equilibrium; hence R must evidently be equivalent to the four parallel forces, in other words R *is the resultant of the four parallel forces and acts through the point* O *obtained by the intersection of the lines* E_3 *and* E_4.

Since the four original forces are parallel, the line $abcde$, on which the magnitudes of these forces are set off, is a straight line, and the length ae gives the magnitude of the resultant R. Further, in the force triangle pae we have represented the three forces shown in Fig. 22 (III), viz. E_3 by ep, E_4 by pa and R by ae.

18. The Link Polygon and the Polar Diagram. The figure 3124 in Fig. 22 (I), which is made up of links, is known as a **Link Polygon.** It is also sometimes called a "funicular polygon" or "cord polygon" from the idea that the links may be formed of "funicles" or

cords. In some cases, as we shall see presently, the link polygon forms a closed figure or polygon, but the word "polygon" is usually applied indiscriminately to both "open" and "closed" link polygons.

The purpose of a **Link** is to provide for the application of two equal and opposite forces between two given forces or between two points at which a number of forces may be acting. A link need not, however, be looked upon merely as a geometrical device. As we shall see, when dealing with framed structures having pinned connections, every member in such a structure may be conceived as a "link", since in each of them two equal and opposite forces act when the members are stressed by loads coming upon the structure; see para. 27.

The line *abcde* in Fig. 22 is known as a **Load Line**. When the parallel forces represented thereon are in equilibrium, then the load line becomes a closed force polygon, though still represented by a single straight line.

The point p in the diagram, Fig. 22 (II), is known as the **Pole.** The lines radiating from p to the load line or load polygon form the **Polar Diagram.**

It should be noted that the load line or load polygon is derived from the external forces acting in the original system, while the polar diagram is derived solely from the link polygon and represents the forces acting in the links.

Problems. In using this method for the solution of problems, the order of procedure is different from that set out in the above explanation. The work should be done in the following manner:

(*a*) Set down the forces on the load line.

(*b*) Select any convenient pole p and draw radiating lines to each point on the load line.

(*c*) Starting at any convenient point on one of the forces represented in the space diagram, draw the links in order, each link being drawn parallel to the corresponding line in the polar diagram.

(*d*) Draw the lines representing the two outer inclined forces and produce them to intersect at the point O, giving a point on the line of action of R, the resultant.

(*e*) The magnitude of the resultant R is then obtained from the load line.

Note. The reader should verify by actual trial that, no matter where p is chosen or where the link polygon is started, the result is the same, the point O found in this way always lying on the line of action of R.

19. Experiment. *To find the resultant of four parallel forces and to demonstrate the relations existing between the link polygon and the polar diagram.*

Using a vertical board, suspend from two pegs at M and N a number of links, as shown in Fig. 23. A spring balance should be inserted in each of the links and the links connected by small wire rings. (The weight

of the apparatus should be allowed for by taking the readings of the spring balances, when no loads are attached, as the "zero" readings.)

From the rings at F, G, H and K suspend four suitable loads.

Mark the exact position taken up by the links and loads by pricking through upon a piece of paper placed behind. Tabulate the magnitudes of the loads and the forces acting in the links. Produce the lines NF and MK to meet in the point O.

Now insert pegs in the rings at F and K and remove the four loads from F, G, H and K.

Fasten a weight R (equal to the sum of the four vertical forces) to a ring and attach also two cords. Pass these cords through the rings at F and K and adjust the lengths of these cords until the ring is over the point O marked on the paper beneath.

On removing the pegs from the rings at F and K there should be no movement in the cords, showing that the vertical force R balances the forces acting through K and F along the lines KM and FN respectively.

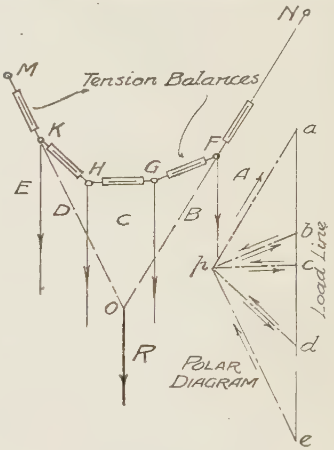


Fig. 23.

Remove the sheet of paper from the board and to a suitable scale set down the load line $abcde$. Starting at point a, draw ap parallel to the line NF. Similarly draw bp parallel to the link FG, cp parallel to GH and so on until all the radiating lines are drawn.

If the experiment has been carefully carried out, the radiating lines will intersect in the point p, and the lengths of the radiating lines will represent to scale the magnitudes of the forces acting in the links.

Note. Each line radiating from p, *with the exception of lines pa and pe*, represents the two forces acting in one of the links; thus pb ("p to b") represents the force acting at point G in the link GF, while bp ("b to p") represents the force acting in the opposite direction in the same link but at the point F (see small arrows on the polar diagram, Fig. 23).

20. Experiment. *To find whether the methods applied to parallel forces also apply to inclined non-concurrent forces.*

Arrange a piece of apparatus similar to that used in the experiment described in para. 19, but with the applied forces AB, BC, CD and DE acting in various directions inclined to one another; see Fig. 24.

When the forces and links have taken up definite positions, mark these positions on a sheet of paper placed behind the links; tabulate the magnitudes of the applied forces and also the forces acting in the links. Produce the lines of action of the two extreme forces MF and NK until they meet in some point O.

Having lettered the spaces between the forces, draw the incomplete force polygon $abcde$, which in this case is not a straight line. This will represent to scale the four known forces acting at F, G, H and K. If we now join the points a and e, then ae will evidently represent the resultant (R) of the four forces. From the similarity of the construction to that adopted in para. 17, evidently O is a point on the line of action

of R; hence a line can be drawn through O parallel to ae to represent this force.

If we now draw the polar diagram, an additional check will be provided. Consider the equilibrium of the point K. Draw bp and pa parallel to KH and KN respectively and so determine point p, the pole of the polar diagram. Complete the polar diagram by drawing radiating lines from p to each of the points c, d and e. Compare the magnitudes thus obtained with the forces obtained experimentally in the three links FG, GH and HK, and also the single forces acting beyond the points F and K.

The results should indicate that the position and magnitude of the resultant of a number of inclined non-concurrent forces may be obtained by this method.

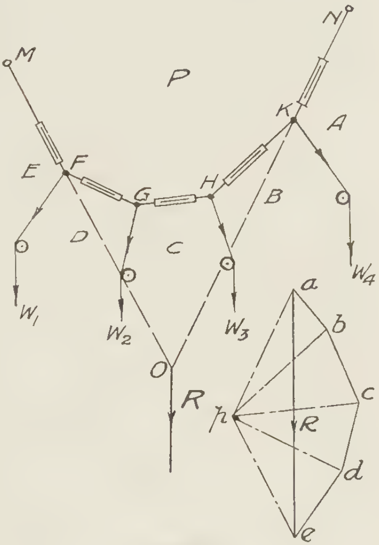


Fig. 24.

Problems. In applying the method to the solution of problems the same procedure should be followed as was outlined in para. 18, except that for "load line" we read "load polygon".

The following example will explain the above points and illustrate an interesting application of the link polygon.

Example. *Forces in a loaded cable. Using the particulars given in Fig.* 12, *find* (a) *the resultant* (R) *of the four forces suspended from the cable;* (b) *the horizontal pull in the cable at points* A *and* B; (c) *the shape which the cable will take up when fully loaded, if the inclination of the cable at* A *is* 30° *as before; and* (d) *the forces acting in each portion of the loaded cable.*

(a) Set out the space diagram as shown in Fig. 25 (A). By means of the load line $lmnst$ and *any* pole p_1, draw the polar diagram and the link polygon $AcdekB$. Produce the lines Ac and Bk to meet in O. Then O lies on the line of action of R. The magnitude of R is given by lt and equals 2000 lbs. The line of action of R is thus shown to be 78 ft. from the vertical through A.

(b) Set out the space diagram once more as at Fig. 25 (B). Draw the line $A_{,}O_{,}$ at 30° to the horizontal to meet the line of action of R in $O_{,}$. Draw $A_{,}B_{,}$ horizontally, then the line $B_{,}O_{,}$ must be the line of action of the cable at B. (The inclination is 47° as in the example in para. 10.) Draw tp_2 and lp_2 parallel to $A_{,}O_{,}$ and $O_{,}B_{,}$ respectively. Then lp_2t is the force triangle for the forces acting at $O_{,}$, and the horizontal line p_2x will give the horizontal component of the pull at *each* end of the cable; the result is 1212 lbs. as before.

(c) Using p_2 as the pole, draw another polar diagram and from it set out the link polygon $A_{,}c'd'e'k'B_{,}$. This gives the shape which the cable will take up when fully loaded. (It also gives to scale the length of the cable and the points at which the loads should be attached.)

(*d*) The forces acting in each portion of the cable—each of which forms a "link"—can be obtained by measuring on the force scale the lengths of the corresponding lines radiating from p_2.

21. The Closed Link Polygon. If we consider once more the conditions set out in Fig. 22, it will be found that, while the line *ep* in the polar diagram represents the force E_3, there is no corre-

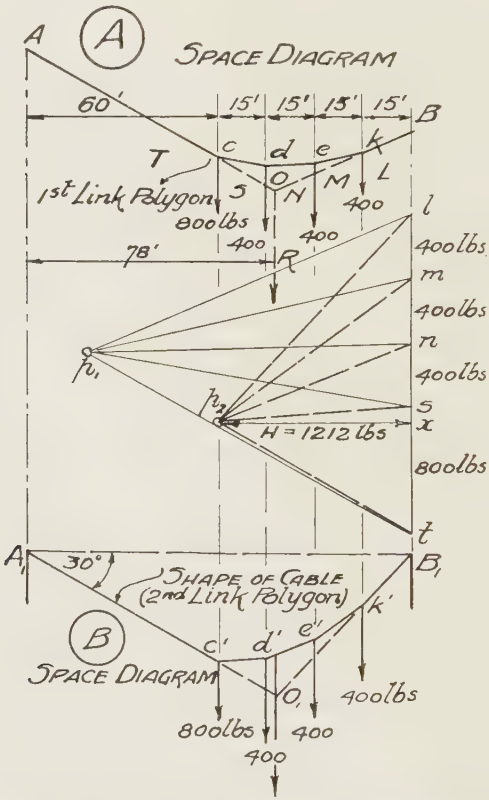


Fig. 25. Forces in and shape of the loaded cable of a concrete-placing plant.

sponding force acting in the opposite direction—as in a link—and represented by *pe*; this is also the case with force E_4 represented by *pa*. From this it should be obvious that the sloping lines E_3 and E_4 do not represent links but merely single forces, these forces being necessary to balance the original system of four parallel forces—*which was not in equilibrium*—together with the system of three links which was added thereto.

If the system of parallel forces had itself been in a state of equilibrium, then such additional single forces as E_3 and E_4 should have been unnecessary *provided that the combined system of links and forces was also in a state of equilibrium.* Let us consider such a case.

In Fig. 26 we have six parallel forces, AB, BC, CD, DE, EF and FA. The first four forces act downwards and the last two upwards. These six forces are known to form a system in equilibrium. If, now, in the manner already described, we impose upon this system an unclosed link polygon $NGHKLM$, it should be obvious from an inspection of the figure that, while each pair of forces acting in the links may be balanced, the complete system so produced is not in equilibrium, for clearly the "frame" made up of the links NG, GH, HK, KL and LM would collapse under the conditions indicated. Some additional factor is necessary. This is still more obvious if we consider the forces acting at N (or M), where we have only the two forces, FA and AP, acting at this end of the link; since these two forces are neither equal nor opposite, they could not alone produce equilibrium.

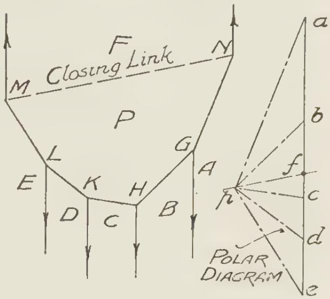


Fig. 26. The closed link polygon.

By adding an additional link (the "closing link" or "closing line") MN, as shown by the dotted line, our link polygon becomes a "closed" figure. From an inspection of the figure or frame so produced, it will be clear that it would be able to resist the action of the six parallel forces without collapsing, in other words the system of links is now in equilibrium. In addition, its introduction has not affected the equilibrium of the system of six parallel forces to which it has been added.

These conditions are made clearer by an examination of the link polygon and polar diagram; see Fig. 26. First add to the polar diagram the line pf, drawn parallel to the closing link MN; then consider the forces acting at the point N (or M), and represented by the sides of the force triangle pfa. The two forces acting in the link NG are evidently tensile forces so that the force acting in this link at N will act away from N; this force is therefore represented on the polar diagram by ap. To prevent collapse of the link polygon the link MN must evidently be in compression, so that the force acting in this link at N must be acting outwards; this force is therefore represented by pf. The remaining force FA acts upwards and clearly must be represented by fa. A similar examina-

tion will show that, at each point in the link polygon through which one of the vertical forces passes, the forces are in equilibrium. It follows from this that a closed link polygon may be added to a system of parallel forces which is in equilibrium without affecting this condition. (By similar reasoning the same may be shown to be true in the case of systems of non-parallel forces.)

22. Conditions of equilibrium. If, in the case illustrated in Fig. 24, we replace R by an opposite and equal force E, the force polygon *abcde* becomes a closed figure and we know that $\Sigma V = 0$ and $\Sigma H = 0$.

Since, however, we are dealing with a system of non-concurrent forces, we cannot be sure that the system is in equilibrium unless the condition $\Sigma M = 0$ is also satisfied; see para. 10.

Let us suppose in this case that the system is not in equilibrium and that ΣM is equal to a moment of say $+ m$.* Then, having added a system of links, the equilibrium of *the combined system* of links and forces will only be assured, if two such forces as E_3 and E_4 act in addition, so as to produce an opposing moment of magnitude $- m$, *that is in such a case the link polygon cannot be a closed figure.* Evidently then the closing of the link polygon is *equivalent* to the condition $\Sigma M = 0$, and we may make the following general statement:

I. In order that a body acted on by a system of coplanar forces may be in equilibrium:

(A) The force polygon must close;
(B) The link polygon must close.

(Compare this statement with that given in para. 10, to which it is equivalent.)

Let us now consider this case experimentally and show, at the same time, how this method may be applied to find the reactions at each end of a structure subjected to a system of parallel forces.

23. Experiment. *Let it be assumed that the magnitudes of the two forces acting at the points* M *and* N *in Fig.* 27 *are unknown. To find these magnitudes both graphically and experimentally.*

Arrange a system of four parallel forces, supported as in the last experiment by a system of links, but having in addition another link MN to form a closed figure; see Fig. 27. (While tension balances will serve to measure the forces in the first set of links, the link MN must contain a compression balance to measure the compressive forces acting in that link. For simplicity the tension balances may be omitted altogether in this and later experiments.)

* In such a case it may be shown that the moment m is produced by a couple, which consists of two parallel forces acting in opposite directions and neither in equilibrium nor reducible to a single resultant; see para. 72.

The whole system of forces and links is then suspended from two tension balances at M and N.

Place a sheet of paper behind the cords and weights and mark off the positions of the links and forces, noting the magnitudes of such forces as are measured experimentally.

Graphical analysis. Having removed the paper, letter the spaces between the various forces. Set off the known vertical forces on a load line ae. Draw the polar diagram and unclosed link polygon as already described.

Since the two supporting forces balance the four downward acting forces and all are vertical, it is clear that the two former must be equal in magnitude to the four latter forces, though they are opposite in direction. The problem therefore resolves itself into dividing ae into two parts which will represent the two upward acting forces.

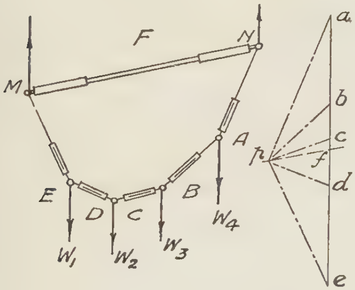


Fig. 27.

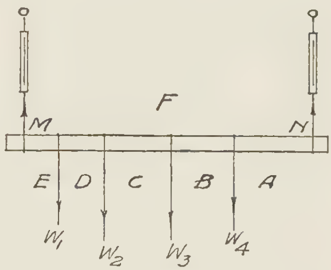


Fig. 28. Reactions of a beam.

If MN be drawn to represent the closing link and the line pf be drawn parallel to this, it follows from a consideration of the equilibrium of the forces acting at N (or at M), as given in the preceding paragraph, that the point f divides the line ae into two portions ef and fa which represent the two forces acting upwards at M and N respectively.

If the link MN be looked upon as a beam—which may be either horizontal or inclined—then the two supporting forces at M and N are the reactions. *Experimentally* the work of finding the magnitudes of these two forces is simplified by attaching the loads to a solid beam as shown in Fig. 28; see Problems IV, 1.

Experiment. If in the Example given in para. 20, see Fig. 25 (B), the line $A_{,}B_{,}$ is drawn, this is evidently the "closing link" of the link polygon $A_{,}c'd'e'k'B_{,}$. Drawing p_2x on the polar diagram parallel to $A_{,}B_{,}$, then the point x should divide the line lt into two parts, so that tx represents the supporting force at A and xl the supporting force at B. By measurement tx gives 700 lbs. and xl gives 1300 lbs. (These values are the same as those obtained in para. 10 by calculation.)

24. The reactions of structures acted upon by inclined forces.

Let four inclined non-concurrent forces be supported by a closed system of links as shown in Fig. 29; we require to know what are the reactions or supporting forces acting at the points M *and* N, *the directions and magnitudes being at present unknown.*

Draw the partial force polygon *abcde*. Then *ae* represents the resultant of the four inclined forces in magnitude and direction. Complete the polar diagram, as already explained, with the exception of the line parallel to the closing link *MN*.

In para. 23 we were able to draw a line *pf* from *p* parallel to the closing line of the link polygon and so obtain the reactions at *M* and *N*, because we knew that these reactions were both vertical and parallel to the resultant; in the present case, however, we do not as yet know in what directions the reactions may act and it is not possible to proceed further without additional particulars.

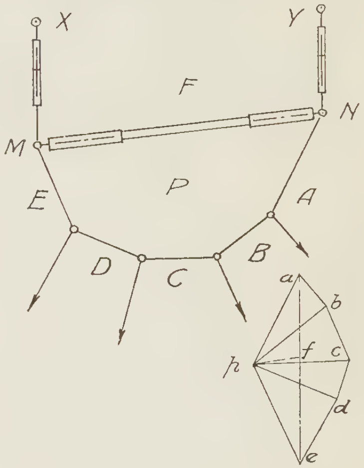


Fig. 29.

There are two possible cases into which problems of this type may be divided, the classification depending upon the manner in which the ends *M* and *N* are supported. The two cases are:

(A) The reactions are parallel to each other;

(B) The reactions are not parallel but the direction of one of them is known.

Case A. *Reactions parallel.* If we refer to Fig. 29, we see that the line *ae* must represent the resultant of the inclined forces. Obviously if the lines *MX* and *NY* along which the supporting forces act are to be parallel to each other they will also be parallel to this line *ae*.

To find the magnitudes of these supporting forces we proceed as in para. 23. Draw the line *pf* through *p* parallel to the link *MN* to cut the line *ae* in *f*. Then, by considering the equilibrium of points *M* and *N* as before, we see that *ef* gives the magnitude of the reaction at *M* and *fa* gives the magnitude of the reaction at *N*.

Case B. *Direction of one of the reactions is known.* A new diagram has been drawn; see Fig. 30. The work can proceed as before up to the point at which we wish to know the directions of the reactions. In this case let us assume that we know that the reaction at *N* acts at right angles to *NM*. Draw *YN* to show the line of action of this supporting force. There are two methods of completing the problem.

(*a*) Consider the forces which are keeping the point *N* in equilibrium. We have the force in the link *NK*, which is represented

on the polar diagram by *ap*. There are also the forces acting in the vertical line NY and in the closing link NM. Through a draw af parallel to NY, and through p draw pf parallel to the link MN. These intersect at f.

Then evidently pf represents the force acting in the link MN at N and fa represents the supporting force acting in the line NY, i.e. the reaction at N.

Consider next the equilibrium of the point M.

From what we have already done we know that fp must represent the force acting in the link MN at M; and we also know that pe represents the force acting in the link ML. Hence, since there are only three forces acting at the point M, the third force must be represented by ef, the third side of the force triangle pef.

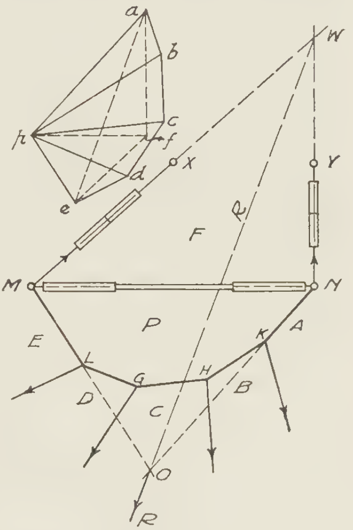


Fig. 30.

Therefore the supporting force or reaction at M is represented in magnitude and direction by the line ef.

(*b*) The reaction at M may also be found as follows. The line ae represents the resultant (R) of the four inclined forces, and together with the two supporting forces ef and fa (which are not yet fully defined) must form the closed force triangle aef. But we know (see Vol. I, Chap. VII) that *if a body is kept in equilibrium by the action of three forces then these forces are concurrent forces and the lines of action of these forces will, if produced, all pass through one point.* Find, in the manner already explained, the point O through which the resultant of the four inclined forces acts, draw the line of action of R and produce it to meet the line of action of the supporting force at N in some point W. Join MW. Then the line MW will give the line of action of the supporting force at M. If ef be then drawn through e parallel to MW, its intersection with af at f will complete the force triangle aef and fully define the two reactions at M and N.

25. Experimental verification. The results of the above discussion may be confirmed experimentally in a manner very similar to that described in earlier experiments. The only new practical difficulty likely to arise will be in arranging the reactions to lie in the desired directions. The details given in the following experiments will indicate how this

may be done. These experiments are specially interesting because they deal with the practical applications in which the above methods are most frequently utilised.

Experiment. Case A. *A body is acted upon by six inclined forces as shown in Fig.* 31, *these forces are known with the exception of the two supporting forces acting at* A *and* B *and these are parallel to each other. To find the magnitude and direction of the supporting forces.*

In order to ensure that the two supporting forces are parallel, the points X and Y, to which the balances are attached, are marked off on a rod at a distance apart equal to AB. The other loads or forces having been attached, the rod XY is moved horizontally until a position is obtained in which all the forces are in equilibrium, the forces along AX and BY being parallel. The weight of the body should be allowed for in the vertical force acting at C.

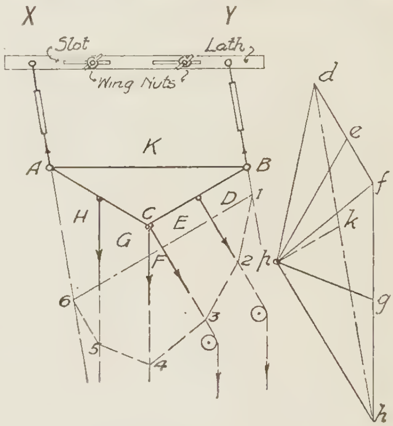


Fig. 31. Reactions of a roof truss—parallel reactions.

(**Note.** The figure to which the forces are attached represents a roof truss but, because of the experimental difficulties which would arise if the attempt were made to arrange the forces to push against the truss, as they would do in practice, all the forces have been reversed. The appropriateness of the application is not affected by this experimental arrangement.)

Having arranged the forces and loads, mark on a sheet of paper the direction of all the loads and tabulate all the forces which can be measured.

Graphical analysis. Remove the paper and proceed as follows: Letter the spaces as shown, not forgetting the space between the two supporting forces at A and B. Draw the force polygon *defgh*. Evidently *hd* is the equilibrant of the four inclined forces. We have already seen that the two reactions, which are to be parallel to each other, should act along lines which are parallel to *hd*; in addition it is evident that these two reactions when added together must be equal in magnitude to that of the equilibrant *hd*. The problem therefore resolves itself into one of dividing *hd* into two parts representing the magnitudes of the reactions. To do this we must make use of the link polygon and the polar diagram.

Since the directions of the reactions at A and B are now known, the link polygon may be conveniently started at any point on the line of action of either of these forces. Having selected such a point, say point 1, the first two links, 1–2 and 2–3, should be drawn *in directions which appear to be suitable*. By drawing lines parallel to these links through the points *d* and *e* on the force polygon, we obtain the position of the pole *p*. The polar diagram can then be drawn in the usual way and the link polygon 123456 completed from it. If *pk* is now drawn parallel to the closing line 6–1, it will divide the line *hd* at *k* so that *hk* represents the magnitude of the reaction at A and *kd* that of the reaction at B.

Occasionally a second and even a third trial may have to be made before a position is obtained for p which gives a clear and well-arranged link polygon. As a rule, however, errors can be avoided, even where the link polygon becomes a complicated figure, if care is taken to make each link cross the correct space, that is the space bearing the same letter as the line in the polar diagram; thus the link 2 3, which crosses the space E, is drawn parallel to the line pe.

Check the reactions at A and B, comparing the magnitudes obtained graphically with those obtained experimentally. Also test whether the lines of action of the reactions lie parallel to the line of action of the equilibrant hd.

Experiment. Case B. *The conditions are as in the last case, but one of the reactions is known to act vertically. To find the magnitude of the two reactions and the direction of one of them.*

It will be found convenient in arranging the experiment to fix the point Y, from which the supporting balance at N is suspended, and to move the point X, from which the other balance is suspended, until the line YN is vertical; see Fig. 32.

Having lettered the spaces as before, draw the incomplete force polygon $defgh$. Obviously we cannot draw any line from h to represent the reaction at M since, as yet, we do not know the direction of this reaction. We do, however, know that the reaction at N acts vertically, so that we can draw a vertical line through d to indicate its direction. To complete the force polygon we must make use of the polar diagram and link polygon.

Observe that though we do not know the direction of the reaction at M we know that its line of action must pass through point M. *In this and similar cases we make use of this fact by commencing to draw the link polygon at the point* M. The reason for this should be fairly obvious, for, if we started the link polygon at N, or in fact at any other point, then, unless the link 1–2 across the space H actually passed through the point M, we could not complete the link polygon.

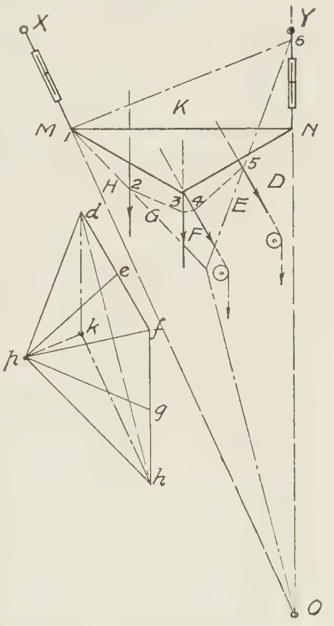


Fig. 32. Reactions of a roof truss—direction of one reaction given.

As in the last case, the first two links 1–2 and 2–3 should be drawn in what appear to be convenient directions; by drawing hp and gp parallel to these links we obtain p and can complete the polar diagram. The link polygon 123456 is then completed. The line pk drawn parallel to the closing line 1–6 will cut dk in the point k. Evidently kd represents the supporting force at N. Join hk and consider the forces acting at the point M. Since ph represents the force acting at M in the link 1–2 and kp represents the force acting in the link 1–6 then, evidently, the third side hk of the force triangle phk represents the direction and magnitude of the reaction at M, which gives the direction of MX.

As before, the values obtained graphically for the direction and magnitude of each of the supporting forces should be compared with those obtained experimentally. An additional check may be made by ascertaining whether the line of action of the resultant dh of the four forces, DE, EF, FG and GH, passes through the point of intersection O of the two reactions at M and N; see Fig. 32.

Problems IV

1. A beam AB is loaded as shown in Fig. A; find the reactions at A and B.

(**Procedure.** Letter the spaces as shown in clockwise order. Set down the load line cf, marking off the loads to scale. Select any point p and draw the polar diagram. Using the lines obtained in this way, draw the

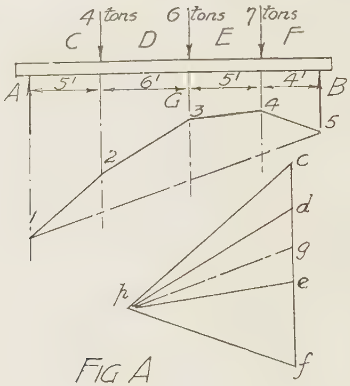


FIG A

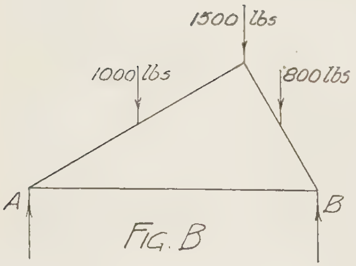


FIG. B

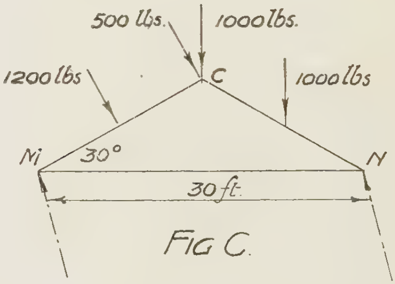


FIG C.

link polygon 12345. Draw the closing line 1–5 and draw the line pg parallel to it through the point p.

Then evidently fg is the reaction at B and gc the reaction at A.)

2. A roof truss is loaded as shown in Fig. B; find the reactions at A and B by means of a closed link polygon. The rafters are inclined at 30° and 60° respectively and the loads occur at their mid-points. (Check the results by arithmetical methods.)

3. Let the forces acting in Fig. 22 be: AB, 3 lbs.; BC, 5 lbs.; CD, 4 lbs.; and DE, 4 lbs. In the same order, let the distances between the forces be $2\frac{1}{2}$ ins., 2 ins. and $1\frac{3}{4}$ ins. Find the magnitude and position of the resultant (R) of the four forces.

4. If in Fig. C, MNC represents a roof truss in which MN is 30 ft. and angles CMN and CNM are both 30°, find the magnitude and directions of the reactions at M and N when the roof loads are as shown. The intermediate loads divide the rafters MC and CN into equal parts.

5. For the same dimensions and loading as in Problem 4, find the reactions when the reaction at M acts vertically.

CHAPTER V

THE FORCES ACTING IN FRAMED STRUCTURES

26. The functions of framed structures. In the erection of buildings it is frequently both necessary and economical to provide certain special structures at convenient positions to carry a concentration of loading such as may arise from the weights of roofs, floors, walls, etc. Familiar examples of such structures—which are usually framed—are roof trusses, trussed beams and girders, trussed partitions, scaffolding and other temporary erections such as arch centres.

The first important step in the design of these trusses is to ascertain the magnitude and nature of the forces acting in each of the members. We have already seen (see Vol. I, Chaps. VI and VII) that it is possible, by the application of certain elementary principles of statics, to analyse the forces acting in the members of a simple roof truss. With the aid of the investigations already made in this volume, we may now proceed to elaborate that treatment and to apply it in the solution of the more difficult problems which occur in larger and more complicated structures.

27. The simple triangular frame.

Experiment. *To find the forces acting in the members of a simple triangular frame when loaded at the apex.*

The apparatus consists of a triangular frame of links supported by two spring balances as shown in Fig. 33 (a), a weight W being attached at C. Thus we have a triangular frame composed of three members freely jointed at the corners. It is assumed that only the force W at C is fully defined, the magnitudes of the other two external forces and the nature of the forces acting in the members being required.

Graphical analysis. To a suitable scale draw the space diagram as in Fig. 33 (b), to represent the frame and the line of action of the load and each of the supporting forces respectively.

Letter and number the external and internal spaces according to the following plan (which will be adhered to throughout this volume): *Letter* all the spaces between the external forces or loads, placing the letters in clockwise order around the truss; *number* all the internal spaces, the order being immaterial; see Bow's Notation in Vol. I.

The load W acting at the point C is obviously supported by two forces acting in the inclined members. To a suitable force scale set down the line ef to represent the load W at C; see Fig. 33 (c). Drawing $f1$ and $1e$, parallel to CA and CB respectively, we have the force triangle $ef1$, which determines the magnitude and direction of each of the inclined forces; it is clear that both these forces act away from the point C, as is shown in Fig. 33 (b).

But, for equilibrium, equal and opposite forces must be acting at the other ends of the sloping members AC and BC, as indicated in Fig. 33 (b).

This additional information completes our knowledge of at least one of the three forces acting at A (and also at B), and so enables us to draw the two further force triangles, *de1* and *fd1*, shown in Fig. 33 (d) and (e) respectively.

Take tracings of the two triangles *ed1* and *df1* and place them over the force triangle *ef1* so that lines *d1* coincide; it will be found (i) that the two triangles *ed1* and *df1* together make up the figure *edf1*, see Fig. 33 (f), and (ii) that in this complete figure each line represents two equal and opposite forces: e.g. the line *d1* represents the force acting in the member AB towards the point A, while the line *1d* represents the equal and opposite force in the same member acting towards the point B.

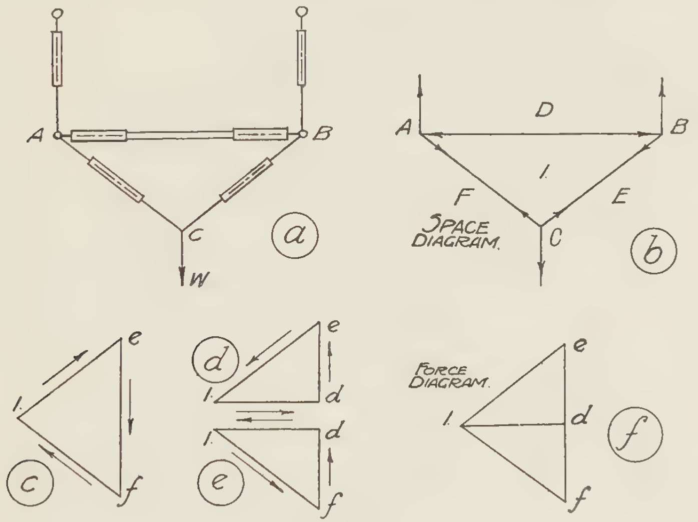


Fig. 33.

The diagram shown in Fig. 33 (f) is known as the **Force** (or Stress) **Diagram.** Since, as we now see, this diagram can be drawn directly from the space diagram, it will not be necessary for us in future to draw separate and independent force diagrams for each of the joints in the truss.

The above discussion indicates that we may look upon the members AB, BC and CA, in the freely jointed frame ABC, as **Links** in the sense defined in the previous chapter, a pair of equal and opposite forces acting in each of the members or links of the frame, the external loads or forces being applied at the intersections of the links. *We shall in fact find it very convenient to look upon such a frame of freely jointed members as an elaborate closed link polygon.*

Further, see Fig. 33 (f), the place of the "pole" in the polar diagram is now taken by the number denoting the space bounded by some (or all) of the members of the frame: e.g. the lines radiating from *1* in Fig. 33 (f) correspond to the forces acting in the links AB, BC and CA which enclose the space marked "*1*" in the Space Diagram. This correspondence will

be found to exist even when the truss becomes a very elaborate and complex structure, in which, of course, the complete force diagram will be made up of many figures similar to *edf1*.

Experimental verification. The comparison of the values obtained experimentally with those obtained by graphical analysis should not occasion any serious difficulty and is left as an exercise for the reader.

28. The importance of the Triangular or Triangulated Frame. The relationship which exists between the space diagram and the force diagram has been compared above with the relationship already seen to exist between the closed link polygon and the polar diagram. There is, however, an important difference to be noted between the link polygon and our triangular frame. If, in the experiment described in para. 23, see also Fig. 27, the magnitudes or directions of any of the four applied forces were altered, then the links in the polygon would take up different positions and

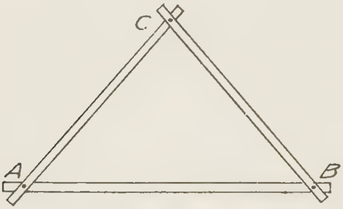


Fig. 34.

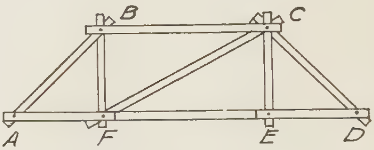


Fig. 35.

the shape of the link polygon would be changed. With a triangular frame, however, such a change in form would not follow a change in the forces acting on the frame, as the following simple tests will serve to demonstrate.

Experiment. Let a simple triangular frame *ABC* be made up as shown in Fig. 34, the links consisting of light wood or metal bars connected together by freely moving pinned joints. It will be found that, within the limits of the strength of the links and joints, it is not possible to alter the shape of the triangle by the application of force at the joints. This characteristic is possessed only by triangular frames.

If a more complex frame be made up of short links, as shown in Fig. 35, since such a figure is made up of a series of triangles, each of which must possess the property of permanency of outline, then it follows that, within the limits of the strength of the members and joints, such a frame cannot be altered in shape by the application of force at any of the joints.

Perfect Frames. Such frames as we have just described, being properly triangulated, are known as **Perfect Frames**; such frames have the exact number of bars necessary to keep them rigid under all systems of loading.

If next a four (or more) sided figure be made up as shown in Fig. 36, with links pinned together at the corners, it will be found that the application of only a slight force, say at *B*, will cause the frame to take

up a new shape such as is suggested by the dotted lines. Similarly if one of the links in the frame shown in Fig. 35 is removed so that one of the spaces enclosed by links ceases to be a triangular space, then the frame ceases to be a perfect frame and is readily altered by the application of a force at any of the joints; see Fig. 37.

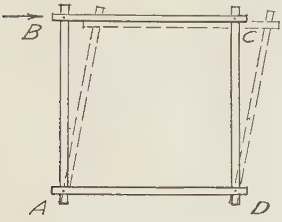


Fig. 36.

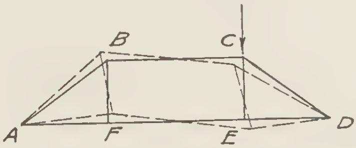


Fig. 37.

Deficient Frames. Such frames as those shown in Figs. 36 and 37 are known as **Deficient Frames**, since they have not a sufficient number of bars to keep them rigid under all systems of loading.

Redundant Frames. If two members AC and BD are added to the four-sided frame shown in Fig. 36, see Fig. 38, there are more links than are necessary to keep the frame rigid because either of the members AC or BD would have been sufficient. Such a frame is known as a **Redundant Frame.**

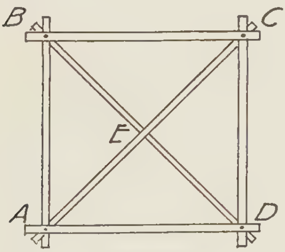


Fig. 38.

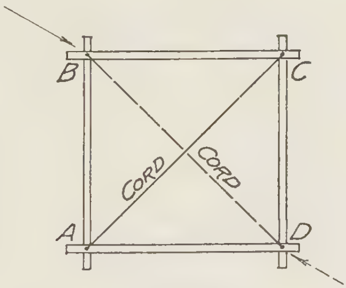


Fig. 39.

Counterbraced Frame. This is a type of frame which, although it may appear at first sight to have a redundant member, may yet be treated as a perfect frame.

Take the four-sided frame $ABCD$ as before and let the joints at A and C and also at B and D be connected diagonally by means of stout cords, chains or thin metal strips; see Fig. 39. If forces be applied at B and D it will be found that, while the cord AC remains tight, cord BD slackens, the first cord supplying the force necessary to maintain the shape of the frame while the cord BD goes "out of action". If the forces be applied at A and C, then these conditions will be reversed. Such a frame is known as a **Counterbraced Frame.**

In practice the diagonal members would be made sufficiently strong to resist any tensile forces which may act upon them, but of such a

cross-section as to bend slightly and so go out of action when subjected to compressive force. In the frame shown in Fig. 38 if the cross links were of light section, *and not connected where they cross at the point* E, then this frame would act as a counterbraced frame.

29. Application of Loads. It is important to notice that in these experiments, and in fact in all the framed structures with which we shall deal at present, the loads or external forces are applied at the joints and not at points intermediate between the joints.

Pinned Joints. In the experiments described above the links forming the frames were connected by pinned joints so that, if not otherwise constrained, the links would be free to rotate about these joints in the plane of the frame. It is obvious that, under such conditions, the links would be subjected to direct forces only, which would be either tensile or compressive forces. If, however, we were to make any or all of the joints into stiff joints, then the forces to which the links would be submitted would be quite different and it would be much more difficult to determine the nature of the forces acting in the links forming the frame.

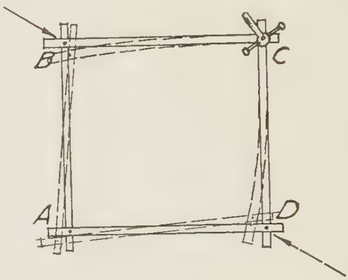


Fig. 40.

Experiment. Use the four-sided frame *ABCD* described above and by means of a bolt or cramp stiffen one of the joints, say that at *C*; see Fig. 40. If now small forces be applied at *B* and *D*, the frame will not be altered in shape; the application of larger forces will, however, cause the links *BC* and *CD* to bend somewhat in the manner suggested by the dotted lines in Fig. 40, provided, of course, that they are not too stout for our present purpose.

In practice stiff joints, redundant members and also members which are continuous over several joints are frequently employed to give added stiffness to a perfect frame or to make a deficient frame rigid, e.g. the member *AFED* in Figs. 35 and 37 is usually a continuous member. From our present point of view the practice is not desirable, since it then becomes very difficult and in some cases almost impossible to ascertain the exact forces acting in the members of such structures. Long experience has, however, justified the use of such devices in well-known forms of trusses, and it is only when an attempt is being made to introduce improvements into such structures, to increase their efficiency or to reduce their cost, that the difficulties involved in the investigation of the forces acting in the members become obvious. Such work is, however,

beyond the scope of this book. We may, therefore, proceed to state what are the assumptions usually adopted in the analytical treatment of the forces acting in framed structures.

I. Unless otherwise stated it will be assumed that the following are the conditions existing in a framed structure:

(i) That the links or members used to build up the structure are connected by pinned joints about which the members may move freely in the plane of the truss;

(ii) That the centre lines of the members and the lines of action of the external loads and forces all pass through the centres of the pins;

(iii) That the members are not continuous over more than two joints.

From the point of view of actual design it can be shown that these assumptions err on the side of safety; their chief value lies in the fact that they enable us to ascertain the forces acting in the members with a considerable measure of accuracy.

Pinned joints may be indicated on the space diagram by a small circle ⊙ round the joint.

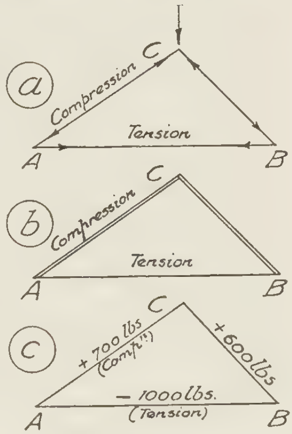


Fig. 41.

30. Struts and Ties. Members of a truss which are subject to tensile forces are known as **Ties**. Members subject to compressive forces are known as **Struts**.

If it is desired to indicate on the space diagram which are struts and which are ties, one of the following conventions may be adopted:

(i) Place arrow-heads near each end of the link, pointing outwards in the case of a strut and inwards in the case of a tie; see Fig. 41 (a);

(ii) Use thick or double lines for struts and thin lines for ties: see Fig. 41 (b);

(iii) Write the magnitude of the force alongside the member preceded by a plus (+) sign in the case of compressive forces and by a minus (−) sign in the case of tensile forces; see Fig. 41 (c).

The most usual practice, however, is to tabulate the members and the nature and magnitude of the forces acting in them. Examples of all these methods are given in the following pages.

We may now proceed to the consideration of a number of trusses of various types. The examples which follow have been

carefully chosen so that each introduces some new point not previously dealt with. It is important therefore that the reader should go through the whole series, together with the problems given at the end of the chapter, in the order given.

31. A king rod roof truss with vertical loading. A simple form of steel roof truss has been chosen for the first example; see Fig. 42. The loading is entirely vertical, the total weight borne by the truss being distributed between the various joints in the manner shown.

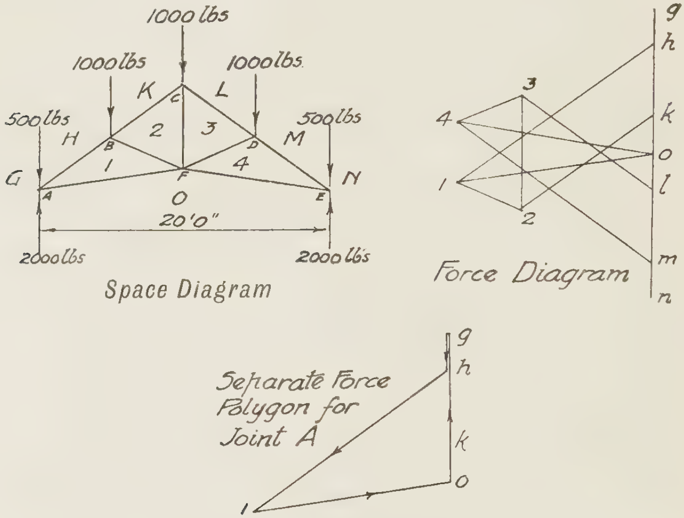


Fig. 42. King rod roof truss.

The analysis of the forces acting in the members should proceed on the following lines:

(i) To a suitable scale draw the space diagram as shown, indicating the magnitude of the external loads and the lines along which they act. In this case, since all the loads are vertical and symmetrically disposed, it follows that the two reactions must each be vertical and equal to half the total load. These reactions may therefore be indicated at once;

(ii) As already described, letter all the external spaces in clockwise order and number all the internal spaces;

(iii) To a suitable scale set down on a vertical line the distances *gh*, *hk*, *kl*, *lm* and *mn* to represent the vertical loads. Next set up from point *n* the distances *no* and *og* to represent the two reactions. Since the two reactions exactly balance the total load, the load polygon *ghklmno* is a closed figure as was to be expected;

(iv) **Force Polygon.** We may now proceed to draw the force polygon for the truss. We usually commence this by considering the forces acting at the end joint *A* (or *E*), since there are only two unknown forces acting at this joint. (The force polygon for this joint has been drawn separately

for the sake of clearness, but this will not usually be necessary once the reader is able to follow each of the smaller force polygons on the complete force diagram.)

Reading the forces around joint A *in clockwise order* commence with the reaction **OG**, which is already known and is represented by *og* on the load polygon. Similarly the force **GH** is represented by *gh*. Of the forces **H1** and **1O** we only know the directions; drawing *h1* through *h* parallel to **H1** and *o1* through *o* parallel to **1O** gives us the point *1*; this completes the polygon *ogh1o* and fully defines the forces acting at A.

To determine the directions in which the forces act we read the large letters in the spaces around joint A in clockwise order, and read the small letters at the corners of the polygon *in the same order*; the direction or sense of a force will then be given by the direction in which we pass from one corner of the polygon to the next. Thus, considering the forces **H1** and **1O**, the lines on the force polygon which represent these forces are to be read "*h* to *1*" and "*1* to *o*". Thus we see that force **H1** acts *towards* joint A and the member is therefore in compression, while force **1O** acts *away* from the same joint and this member is therefore in tension. (**Note.** The direction in which we pass from corner to corner of the force polygon *ogh1o* has been indicated by arrows on the separate force polygon for the joint; see Fig. 42.)

We may now proceed on the main force diagram to consider the forces acting at the joint B; of these we already know, **1H** and **HK**, the other two are as yet unknown except for their directions. Through *k* draw *k2* parallel to **K2** and through *1* draw *12* parallel to **12** on the space diagram, these give point *2* and complete the force polygon *1hk2* for the forces at the joint B. The direction or sense of the forces may be obtained as already explained.

Joint C is taken next. The force in **2K** is represented by the line *2k* on the force diagram and the force in **KL** by the line *kl*. We complete the polygon by drawing *l3* through *l* parallel to the member **L3** and *23* through *2* parallel to the member **23**. The force polygon for joint C is then *2kl3*.

Since the truss is symmetrical in construction and is symmetrically loaded it follows that the force diagram will also be symmetrical, forces in the corresponding members on each side being equal in magnitude and similar in nature. It will not therefore be necessary in such cases to complete more than half the force diagram once the reader has become familiar with the general method. In this case the whole of the force diagram has been given. The magnitudes of the forces may be written alongside the members or may be tabulated in the form shown in the example given in para. 34 below.

Notes. (*a*) The order in which the joints are considered is frequently very important. If there are more than two unknown forces at any joint it will not be possible to complete the force polygon for that joint, and work on some other joint must usually be taken first. For example if, after analysing the forces acting at joint B, we had proceeded to consider joint F, it would have been found impossible to complete the force polygon for that joint, since of the five forces acting at that joint only two were completely defined at that stage. By taking joint C before joint F we were able to find the magnitude and sense of the force in the member *23* and thus define a third force acting at the joint F.

(*b*) We may note from the completed diagram that, if the external load GH and also the load MN were omitted from the force diagram they would reduce the total reactions but *would not affect the forces acting in the members of the truss. This is always the case when the end load acts in the same line as the reaction at that point.* In such cases in future we may therefore omit these loads, at the same time noting that in all those cases where the effect is doubtful it is better to include them.

32. Experiment. The experimental verification of the forces acting in a roof truss is not easily carried out; this is mainly so because of the difficulty of arranging for a series of links which are capable of measuring compressive forces. Links subject to tensile forces are more readily fitted up.* By omitting the measurement of the compressive forces, however, the construction of the apparatus can be simplified. The experimental truss shown in Fig. 43 has been arranged on these lines. (An alternative arrangement is shown in Fig. 45.)

The links AB, BC, CD, DE, BF and DF all being in compression are formed out of pairs of light wood or metal bars, with holes drilled

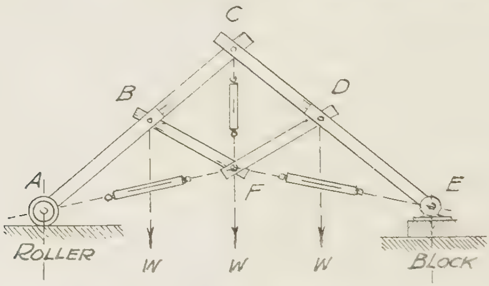


Fig. 43. Experimental roof truss.

at each end. In these holes smooth steel pins are fitted forming a simple pinned joint. The remaining links, being subjected to tensile forces, are made up of light spring balances hooked at one end to the pins already mentioned and at the other end to light brass chains; these chains supply a convenient mode of adjusting the length of the link to take up the strain in the spring balance.

The pins at the two outer joints A and E are made to project about two inches on each side and pass through smooth rollers, which rest upon the bench upon which the experiment is being carried out; these rollers enable the feet of the truss to take up positions conditioned only by the forces acting at the lower joints of the truss and unaffected by friction. For the sake of increased stability the rollers may be omitted at one side and their place taken by a plain block with upright perforated brackets; see Fig. 43.

The loads are put on the truss by suspending weights from the joints B, C and D. The lengths of the tension members are then adjusted until the links forming the upper sloping members *lie as nearly as possible in straight lines.* A graphical analysis is then made of the forces acting in the truss and the magnitudes of the forces thus obtained compared with those obtained experimentally. (**Note.** The readings of the balances when the truss is unloaded may be taken to be the "zero" readings.

* But see type of apparatus shown in Fig. 157 A.

33. Roof Loads. It will be convenient to add at this stage a note upon the nature and magnitude of the loads which come upon roof trusses. These loads are of two kinds: (*a*) **Dead Loads**, that is loads which remain constant, such as those due to the weight of roof timbers, roof covering, etc.; and (*b*) **Live Loads**, that is loads which are constantly changing in magnitude; in the case of roofs this type of load is practically limited to wind pressure.

Dead Loads. Table I gives the weights—calculated for each square foot of roof surface—of a number of roofing materials, etc. These are average figures and should not be used if more accurate figures are available.

Table I

Average weights of roofing materials and framework (lbs. per sq. ft.)

Boarding, 1 in. ...	3	Asbestos corrugated sheets ...	$5\frac{1}{2}$
Slating	9	Corrugated iron	3
Tiling	14	Lead	7
Slating or tiling battens	$\frac{3}{4}$	Timber framing (rafters, etc.)	3
Glazing and bars ...	5	Ceilings	12
Asbestos slates ...	4	Snow	3

Wind Pressure. Assumptions. For the purpose of designing structures sufficiently strong to resist the pressure effects of the wind, it is usual to assume: (*a*) *that this pressure acts at right angles to the surface exposed to the effects of the wind;* (*b*) *that its intensity varies with the inclination of that surface to the horizontal,* a vertical surface thus being subjected to a pressure of greater intensity than an inclined surface; and (*c*) *that the pressure acts as a uniform pressure over the whole of the surface.* The values given below in Table II may be accepted as being in accordance with the general practice in this country, though much difference of opinion exists as to the actual effects of the wind. The figures generally employed are probably much higher than they need be.

Table II

Wind pressures on plane surfaces (lbs. per sq. ft.)

Vertical surface...	30
Surface inclined at 60° or more	30
,, ,, 45° ...	28
,, ,, 30° ...	24
,, ,, 15° ...	15

(**Note.** The above values have been expressed in terms of dead load.)

Distribution of roof loads. Roof trusses are usually spaced at regular distances apart—called "bays"—and the common rule is to assume that each truss carries that portion of the roof surface which extends over half the bay on each side of the truss. The total weight so obtained is then distributed between the joints or "load-points" according to their spacing; see para. 34 below. The wind load is treated in the same manner, though of course it can only act on one side of the truss at a time; see para. 36.

It is usual to add the weight of the truss itself to the total load to be carried. The approximate weight of the truss may be calculated from the following table.

Table III

Approximate weight of trusses (all at 10 ft. centres)*

Timber trusses
Weight in lbs. = 0·9 (span of truss in feet)^2

Steel trusses
Weight in lbs. = 0·64 (span)^2

Composite trusses (timber and steel)
Weight in lbs. = 0·77 (span)^2

34. An unsymmetrical truss with vertical loading. The truss shown in outline in Fig. 44 is used in the construction of factory roofs where good lighting without direct sunlight is required, the steeper surface usually facing north and being glazed. In this case the span is assumed to be 20 ft., the trusses being spaced at 10 ft. centres.

Loading. The longer surface measures 17·5 ft. approx. from top to bottom, hence the total area carried by this side of the truss is ($17·5 \times 10$) or 175 sq. ft. Similarly the area on the short side is (10×10) or 100 sq. ft. If the weight of the roofing materials on the long side averages say 15 lbs. per sq. ft., then the total load will be (175×15) or 2600 lbs. Similarly if the weight on the short side—which is covered almost entirely by glazing averages 7 lbs. per sq. ft., then the total weight is (7×100) or 700 lbs.

These loads are carried at the load-points S, T, U, V, W; see Fig. 44. Since each side is bisected by a load-point, half of each load will be carried at T and V respectively. At the top and bottom points one-quarter of each load will be carried. The totals may be set out as follows:

Load-points	S (lbs.)	T (lbs.)	U (lbs.)	V (lbs.)	W (lbs.)
Left side	650	1300	650	—	—
Right side	—	—	175	350	175
Totals	(650)	1300	825	350	(175)

Since the loads at S and W (in brackets) rest directly on the walls, they can be ignored in finding the stresses in the members of the truss.

Reactions. In this case before we can complete the load polygon we must ascertain the magnitudes of the two reactions. This can be done either (*a*) by calculation, or (*b*) by graphical methods. Proceeding by the latter method we set down the load line *abcd*. Selecting a suitable pole p (preferably to the right of the load line so as to clear the force diagram), the polar diagram is drawn by connecting p to the points a, b, c and d. For convenience the lines of action of the external loads and supporting forces should be produced to extend either below or above the space diagram and the space letters repeated as shown in

* See *Architectural Building Construction*, Vol. II, Part I, by Jaggard and Drury.

small circles in Fig. 44. The link polygon may now be drawn, commencing with link *mn* drawn parallel to line *ap* across the space (*A*). When the final point *r* has been obtained in this way, the closing line *mr* may be drawn. The line *pe*, drawn from the pole *p* parallel to *mr*, gives the point *e* which divides the total load, represented by the load line *ad*, into two parts such that *de* represents the reaction at *W* and *ea* represents the reaction at *S*. The load line thus becomes a closed load polygon.

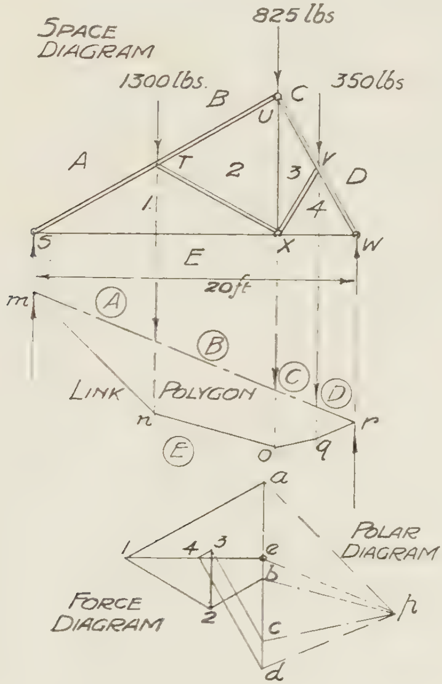


Fig. 44.

Force Diagram. The procedure is similar to that described in the preceding example. The polygons corresponding to the various joints may be set out as follows:

The forces at	joint	*S*	are given by	polygon	*ea1*
,,	,,	*T*	,,	,,	*1ab2*
,,	,,	*U*	,,	,,	*2bc3*
,,	,,	*V*	,,	,,	*3cd4*
,,	,,	*W*	,,	,,	*4de*
,,	,,	*X*	,,	,,	*e1234*

It may be noted that in this case the points *1* and *4* fall on the same line but do not coincide; this indicates that the forces in the two horizontal members are similar but of differing magnitude.

Tabulation of forces. (*a*) The nature of the force acting in each member has been indicated on the space diagram in this example by the use of double lines to represent struts and single lines to represent ties.

(*b*) In addition the complete definition of each of the forces is given in tabular form below.

Struts		Ties	
Member	Force (lbs.)	Member	Force (lbs.)
A–1	2070	*1–E*	1800
B–2	800	*4–E*	870
C–3	1400	*2–3*	800
D–4	1730		
1–2	1270		
4–3	200		

Experimental verification. An alternative method of arranging a roof truss for experimental verification is shown in Fig. 45. All the external loads and forces are reversed in sense with the result that the forces acting in the members of the truss are also reversed, struts becoming

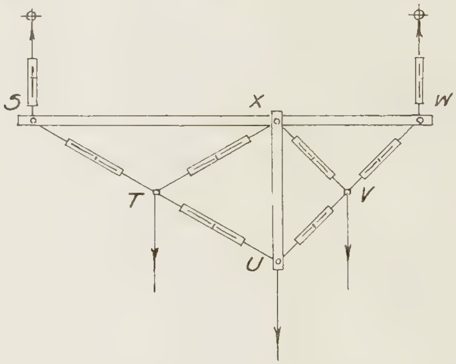


Fig. 45. Experimental roof truss.

ties and *vice versa*. It is then possible to measure more of the internal forces than was the case in the experiment described in para. 32. The links and connections are made up as before explained.

The members SX, XU and WX are first fixed in length, the other members are then arranged in position and "zero readings" taken. After the loads have been applied, links ST and VW are adjusted in length until the link UX is vertical. Links TX and VX can then be adjusted until STU and UVW form straight lines. The graphical analysis and experimental verification proceeds as before.

35. A symmetrically loaded truss with ceiling loads. The form of truss shown in Fig. 46 is most frequently constructed in steel. In the example selected it is assumed that the under-side of the truss is covered by a ceiling which is suspended from the lower joints of the truss.

The truss and all the loading being quite symmetrical, the calculation of the reactions is a simple matter. The chief difficulty in this case arises when arranging the load polygon. As will be seen from the following explanation *it is necessary in this case to take the space letters in clockwise order and to set out the sides of the load polygon in strict rotation.*

Load Polygon. Since all the loads are vertical, the load polygon must be a straight line; in order, however, to make the procedure clear the parts of this straight line polygon have been drawn separately at the side of the load polygon proper; see Fig. 46.

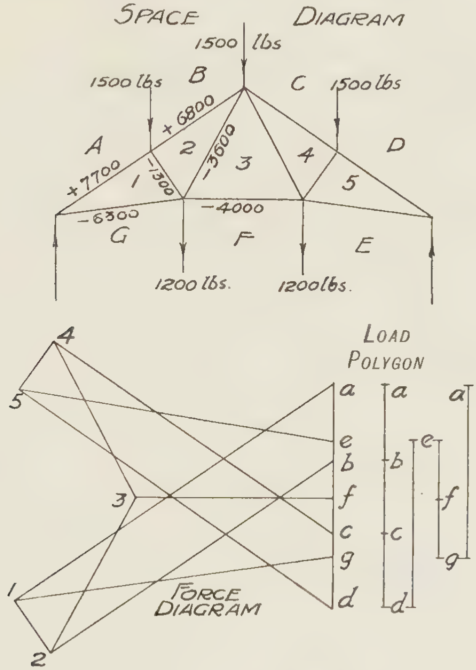


Fig. 46. Roof truss with ceiling loads.

Draw the line *ad* and on it set off to a suitable force scale the loads *AB*, *BC* and *CD*; these are represented by *ab*, *bc* and *cd* all acting downwards. Since the force *DE* acts in an upward direction, set *de* upwards to represent it. The next two forces act downwards and are represented by *ef* and *fg*. Finally the remaining reaction *GA*, acting upwards, is represented by the line *ga*, which closes the polygon.

Force Diagram. The force diagram does not involve any new point and we may proceed as in earlier examples.

Tabulation of forces in members. In this example the magnitude and nature of the forces acting in the members may be indicated as shown in Fig. 46, by writing the magnitude of the force alongside the member preceded by a plus (+) sign if the member is in compression, or by a minus (−) sign if it is in tension.

Experimental verification. The apparatus for experimental work on this truss is most easily arranged on the lines described in the preceding experiment—in which the truss was inverted—and does not present any new difficulty.

36. King post roof truss subjected to wind pressure. The truss shown in Fig. 47 is most frequently constructed in timber. It may, however, be constructed of timber and steel combined or of steel alone. In the present example the vertical loads due to the weight of the roof covering are not considered, the only loads which are dealt with are those due to

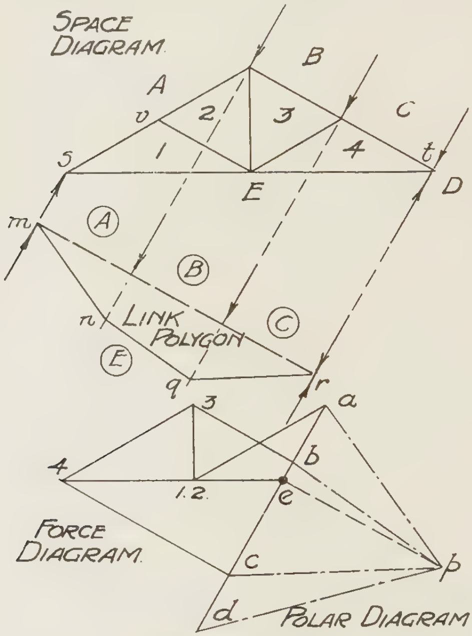


Fig. 47.

the pressure of the wind acting upon the right-hand surface of the roof. As already indicated, the resulting pressure is assumed to act at right angles to the surface affected and to be distributed uniformly over that surface. This loading is transferred to the joints of the truss as shown; see Fig. 47.

Reactions. With trusses of the dimensions usual for this type it is not necessary to adopt any special method of fixing the ends of the truss to the walls, the truss merely resting upon or being lightly secured to a stone or metal template at each end. Under such conditions it is reasonable to assume that each wall will offer approximately equal resistance to any inclined loads acting upon the truss. On this assumption the reactions are taken to act along lines which are parallel to each other and to the line of action of the inclined loads (or to their resultant if these inclined loads are not all parallel).

Load Polygon. Draw the line *ad*, see Fig. 47, parallel to the line of action of the wind loads and set off to a suitable scale the distances *ab*, *bc* and *cd* to represent these loads. So as to keep clear of the space diagram, produce the lines of action of the wind loads below the truss and repeat the space lettering as before; see Fig. 47. It should be noted that, since the line of action of the load *CD* lies in the same line as the reaction *DE*, it will not affect the forces acting in the roof truss but merely increase the total reaction at point *t*. We may therefore, if we wish, leave the force *CD* out of our considerations for the present, dropping also the space letter *D*.

Selecting any suitable pole *p*, draw the lines of the polar diagram *pa*, *pb* and *pc*. The link polygon may be conveniently started at point *m*, from which we draw *mn* parallel to *pa* across space (*A*), *nq* across space (*B*) parallel to *pb*, etc. The closing line is *rm*; the line *pe* is drawn parallel to this, giving the point *e* on the load line. Then *ea* represents the reaction at *s* and the reaction at the other end of the truss is represented by *ce*. If the load *CD* is taken into account, the total reaction at this end is represented by *de*.

Force Diagram. The force diagram may now be drawn in the usual way, commencing at either of the lower joints. If the joint at *s* be selected as the starting-point, we have *a1* parallel to the member ***A1*** and *1e* parallel to the member ***1E***. This completes the polygon for this joint. At joint *v*, since no new load is introduced, we have only to ascertain the forces acting in the three members meeting at that point. We already know that the force in ***1A*** is represented by the line *1a*. Since the member ***A2*** lies in the same straight line, the line *a2* representing it must similarly lie on the line *a1*, the only possible solution to this condition is for the points *1* and *2* to coincide. We may thus conclude that there is no force acting in the member **21** due to this wind load and also that the forces in the members ***1A*** and ***A2*** are equal and similar. The completion of the force diagram should not cause further difficulty.

Experimental verification. A complete verification with a truss consisting of separate members could be arranged but would involve a somewhat complicated piece of apparatus. An experiment dealing with the external forces only in a similar case has already been described; see para. 25.

37. Queen rod type of roof truss, with both wind and dead loads. The type of truss shown in Fig. 48 (I) is often constructed in steel for spans up to 60 ft. It is also suitable for construction in timber and steel. The vertical loads due to the weight of the roof covering are spaced at equal distances apart. It is assumed in this example that wind pressure is acting on one side of the roof.

Reactions. With trusses of very large span (80 ft. or more) it is often advisable to fix one end of the truss to its support while the other end rests upon rollers or is otherwise free to move horizontally when the truss expands or contracts in length owing to changes in load or in temperature. Under these conditions, since it would not be possible to transmit an inclined force through the rollers, it is usual to assume that the reaction at the end where rollers are used is a vertical one. The other reaction must obviously be of sufficient magnitude and inclination to maintain equilibrium.

Load Polygon. Letter the spaces between the external forces and number the internal spaces in the usual way. Selecting a suitable force scale, proceed to draw the load polygon as shown in Fig. 48 (II). Taking the letters and forces in clockwise order and commencing with force AB, we have four vertical forces represented by the lines ab, bc, cd and de. The remaining forces are then alternately inclined and vertical and this portion of the load polygon finishes at point n. The vertical reaction NO is the next force taken in order and of this we know only the direction and therefore draw a vertical line through n to represent it. Before we can complete the load polygon we must know at least the direction of the other reaction OA. This we may now proceed to obtain, using the methods described in paras. 24 and 25.

In this example there are a large number of separate loads to be dealt with and unless these are reduced in number the link polygon becomes very complicated and may adversely affect the accuracy of the results. We can simplify this part of the work by noting the following points: (a) since all the dead loads are symmetrically applied we may replace them for our present purpose by a single force (W_1) acting at the centre of the truss and equal in magnitude to their sum; (b) so far as the windward side of the truss is concerned the loads due to wind pressure are also symmetrically disposed and may be replaced by a single load (W_2) acting at the centre of the sloping surface of the roof and equal in amount to the sum of these loads. These two forces are defined in Fig. 48 (I) by space letters (A), (Q) and (N). The magnitudes of these forces may be readily obtained on the load polygon by setting down first all the vertical loads, obtaining line aq, see Fig. 48 (II); this represents the force (A) (Q), the resultant of all the vertical loads. Next set off similarly the inclined loads obtaining line qn, which represents the force (Q)-(N). As before NO will represent the direction of the reaction at the free end of the truss. Select a suitable pole p and draw the polar diagram consisting of the lines pa, pq and pn. *Start to draw the link polygon from the point* m, *this point being the only point which we know to lie on the line of action of the reaction at* m. The link mn is drawn across the space (A) parallel to the line pa. The link nr is then drawn across the space (Q) parallel to the line pq. The next link rs is drawn parallel to the line pn across the space (N). Then the line ms is the closing line and if the line po be drawn through p parallel to this line then no represents the magnitude of the reaction at the free end and, joining o to a, we must have oa representing the direction and magnitude of the reaction at the fixed end, the load polygon now being closed.

Reactions. The following is an alternative method of finding the reactions. If on the load polygon, Fig. 48 (II), we were to join the points a and n, then the line an would represent, in magnitude and direction, the resultant of all the external loads. We could then consider the truss as being held in a state of equilibrium under the action of three forces, this resultant (R) and the two unknown reactions. The problem would thus be reduced to the one dealt with in para. 24; the truss being in equilibrium under the action of three coplanar forces, then the three forces must be concurrent.

For the sake of clearness a new and simplified diagram has been drawn; see Fig. 48 (III). On this diagram produce the lines (A)-(Q) and (Q)-(N) until they meet in the point x. This point must lie on the line of action of their resultant R, which we have just found to be represented in magnitude and direction by the line an. Through x draw the line of

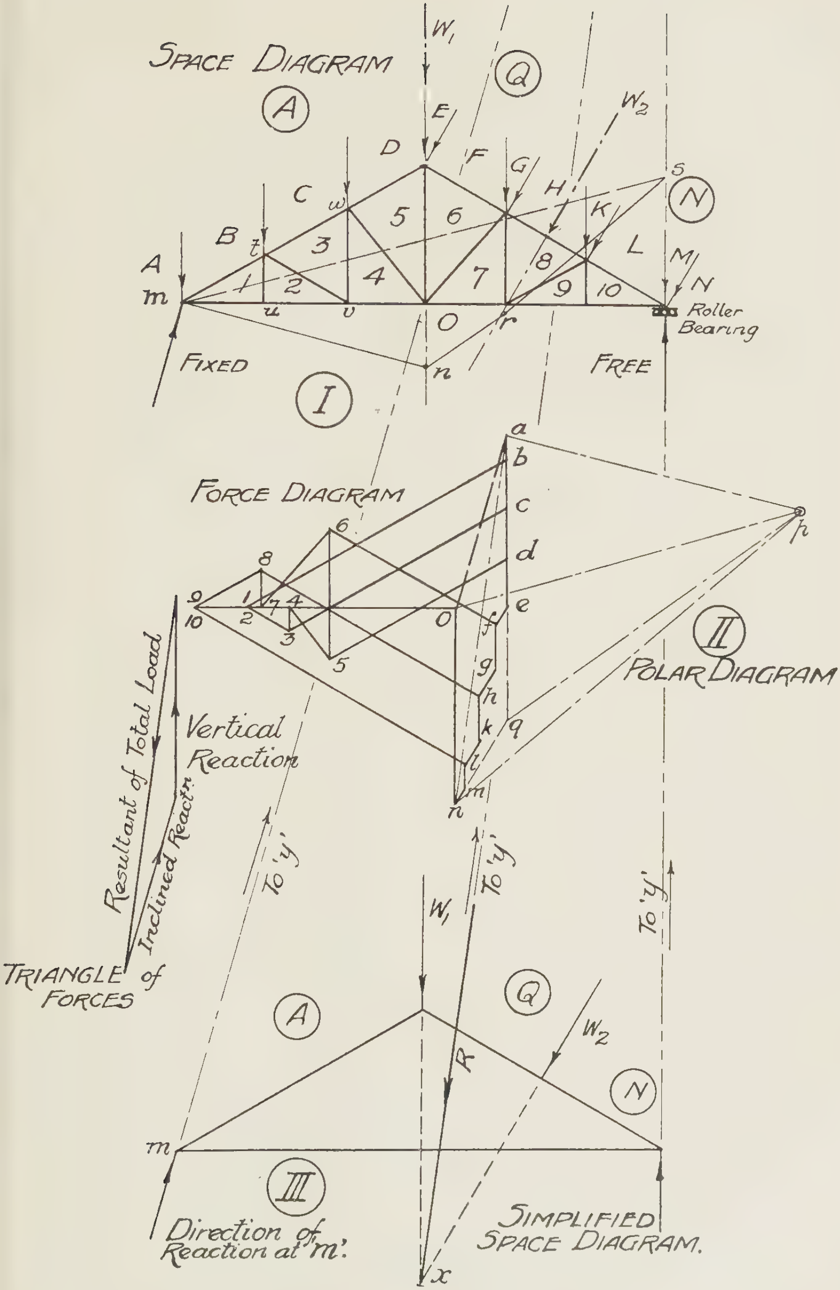


Fig. 48. Queen rod roof truss with wind and dead load.

action of R parallel to an and produce it to intersect, at y, the line of action of the vertical reaction. If the remaining force, the reaction at m, is to be concurrent with these two forces, then its line of action must also pass through the point y. Join m to y, this evidently gives the line of action of the remaining force. The values of the two reactions can now be found by a simple application of the triangle of forces, the magnitude of the total load being already known; see Fig. 48 (III).

Force Diagram. See Fig. 48. Provided that the joints are dealt with in proper order this diagram is not difficult to construct.

38. Maximum forces acting in trusses subjected to wind and dead loads. In the design of trusses it is necessary that full knowledge should be obtained of the maximum—and, in some methods of design, the minimum—forces acting in each of the members of the truss. Since the force due to wind pressure will vary in each member according to which side the wind is acting upon and also according to the way in which the truss is supported, it is sometimes necessary to make a fuller investigation than that described above. This investigation is usually carried out on the following lines:

(*a*) A force diagram is drawn for the truss with dead loads only and the results tabulated in the manner shown below;

(*b*) A force diagram is next drawn for the truss with the wind acting on that side of the truss having a fixed support and the results tabulated;

(*c*) A force diagram is finally drawn for the truss with the wind acting on the free side of the truss and the results tabulated;

(*d*) The maximum and minimum forces acting in each member in the truss are then obtained by finding the algebraic sum of the forces acting in them as the wind pressure acts on one side or the other. This work may be conveniently set out in tabular form as indicated in the following example.

Member	Force due to dead load	Force due to wind on right	Force due to wind on left	Max. force	Min. force
ST	1000 lbs. (comp.)	200 lbs. (tension)	600 lbs. (comp.)	1600 lbs. (comp.)	800 lbs. (comp.)

Except in very large and important structures, such as large bridges, which are beyond the scope of this volume, the members of a truss are usually designed to carry the maximum load (dead load plus live or wind load) coming upon it as though this load consisted entirely of dead load. In such cases the above method, which furnishes very complete information as to the changes taking place in the forces acting in the members, may be simplified and only two force diagrams drawn out. Both of these

diagrams would deal with combined loads, but in one the wind pressure would act on the right and in the other on the left side of the truss.

In the case where the truss is considered to be fixed at both sides, and the reactions are parallel to each other and to the total resultant of the external loads, it will only be necessary to draw one combined diagram for the wind and dead loads, the members all being designed for the maximum forces which are ascertained in this way, opposite members in symmetrical trusses being made of equal strength.

Maximum reactions. It is only for the largest trusses that roller bearings are used. A plan adopted with trusses of moderate span—and even on large spans in the interests of economy—is to fix the trusses to stone or metal templates by means of bolts passing through slotted holes; these provide for a certain amount of lateral movement. Under such conditions it is practically impossible to decide what will be the lines along which the reactions will act where inclined forces are brought to bear on the truss. The conditions are intermediate between those in the two cases described in paras. 36 and 37, and it is usually safest to assume in these cases that the reactions will act so as to bring the maximum forces to bear upon the supporting walls and in the members of the truss. We therefore assume that in most cases the conditions are as stated in para. 37 (one reaction vertical), because it is under these conditions that, other things being equal, we get the maximum of horizontal thrust on the walls (at the fixed end) and also the maximum forces acting in the members.

The reader should work out a case, such as that described in para. 37, and endeavour to satisfy himself that the following statement is true: That where a truss is fixed at one end and free at the other, and is subjected to wind pressure, maximum forces will act in the members (on one or the other side of the centre line) when the fixed end is on the windward side.

39. Framed Girders. Framed girders, such as may be used to carry large platforms, floors, flat roofs, etc., because of their regularity of outline and design, lend themselves to other methods of treatment and will therefore be dealt with at greater length at a later stage; see Chap. XI. They may, however, be dealt with as here explained. As no new difficulties are involved, this work is left as an exercise for the reader.

40. Floor Loads. The loads coming upon floors, in addition to their own weight, are made up of dead loads and live loads, but in the design of floors it is found most satisfactory to design for an

equivalent uniform load covering the whole surface of the floor, except where heavy concentrated loads have to be provided for. Suitable values for general cases are indicated in the following table.

Table IV

Floor Loads (to be added to the weight of the floor structure)

Type of floor	Load per super. ft. (lbs.)
Domestic floors, bedrooms	50
,, ,, living rooms ...	70
Offices	100
Ordinary workshops and retail shops	112
Warehouses	224

Problems V

Note. The methods described in this chapter are to some extent self-checking; thus if the force lines completing a force diagram are not concurrent, that is they do not intersect at a point, then, *either* the diagram has been inaccurately drawn, *or* some error has been made. Such defects can be corrected by the reader if this chapter has been carefully studied. It is therefore suggested that any of the examples explained in this chapter may have dimensions and loads added to them and thus form suitable cases for detailed investigation.

CHAPTER VI

THE EQUILIBRIUM OF SYSTEMS OF PARALLEL FORCES WHICH ARE NOT COPLANAR. THE STABILITY OF CRANES AND SCAFFOLDING

41. The stability of complete structures. In the preceding chapters we have dealt with systems of forces acting upon rigid bodies or frames, in which both the forces and the frames were contained in one and the same plane (coplanar). All buildings are, however, three-dimensional, that is they have length, breadth and height, and in dealing with their design and erection problems frequently arise which involve the consideration of the equilibrium of forces acting in more than one plane—**non-coplanar forces.** These problems need not as a rule receive much consideration in the case of ordinary brick or stone structures because, owing to the comparative independence of the units with which they are constructed, it is usually sufficient to satisfy the conditions of equilibrium in each of the main planes into which the structure may be divided.

In all other cases it is necessary to give careful consideration to the stability of the building as a whole; in the case of steel and timber framed structures, because of the comparative rigidity of the connections or of the method of framing the structure together; in the case of reinforced concrete structures, because of the monolithic or "one-piece" character of the construction; and, in the case of buildings which include large domes, arches or intersecting vaults, because it is important to ensure in such cases that the settlement of the building—which always takes place during its erection, owing to the compression of the stratum upon which it is built—shall take place uniformly over the whole site, so as to avoid cracking and even, in some cases, failure of the parts of the structure.

While the problems arising in such work are in the main beyond the scope of this volume, it is important that everyone responsible for the design and erection of buildings should have some acquaintance at least with the more elementary statical principles which are involved. Fortunately for our purpose, methods may be adopted in the first stages of the work which are not too difficult of comprehension, and they can be largely developed on the basis of the work already treated in this volume. As some comparatively simple and important applications of these methods occur in connection with the consideration of the stability of the larger forms of

scaffolding, hoisting appliances and other large temporary structures used by the modern builder, the importance of which cannot well be over-estimated, it is proposed to utilise them to form the basis of an elementary discussion of these problems.

I. **Stability.** **The stability of a building, that is its power to resist overturning or collapse,** depends upon a series of conditions which may be divided into three groups:

(1) **The external forces acting upon the structure as a whole must balance.**

The forces to be considered are:

(*a*) The weight of the structure;
(*b*) The loads it is called upon to carry (both dead and live loads);
(*c*) The supporting forces, due usually to the reactions or resistances of the foundations.

(2) **Each portion of the structure,** such as a wall, a frame or a floor, **must be capable of resisting the forces brought to bear upon it, and these forces must be in equilibrium.**

(3) **Each separate piece or member of the structure must likewise be capable of resisting the forces acting upon it, and these forces must be in equilibrium.**

The first group is investigated in the next two chapters, though Chap. VII also includes portions of the other two groups, in so far as they may be involved in the consideration of the stability of simple structures.

It will be convenient to deal in the first place with systems of parallel, non-coplanar forces, upon a knowledge of which most of the simpler applications depend. In studying these questions, knowledge of the methods adopted in solid geometry will be found of great assistance.

42. To find the resultant of a number of parallel forces not acting in one plane.

(Method I.) *By solution in successive planes.* Three forces A, B and C are shown pictorially in Fig. 49 (A), all acting in an upward direction and at right angles to the horizontal plane XOZ. The positions of these forces are defined by the points a, b and c, at which their lines of action penetrate the horizontal plane XOZ. It is required to find the resultant of these three forces.

Consider the forces A and B. These act in the same vertical plane, that is the plane which is defined by the three lines Aa, Bb and ab. By applying the methods explained in Chap III, it will be possible to replace the forces A and B by a single force D,

($= A + B$), which acts at the point d, where d is so fixed that we have:

$$\frac{A}{B} = \frac{bd}{ad}.$$

Similarly, in the plane of the two forces C and D, a single force R, ($= C + D$), can be found which will replace these two forces, the position of the point e, which lies in the line of action of the force R, being so placed that: $\frac{C}{D} = \frac{de}{ce}$.

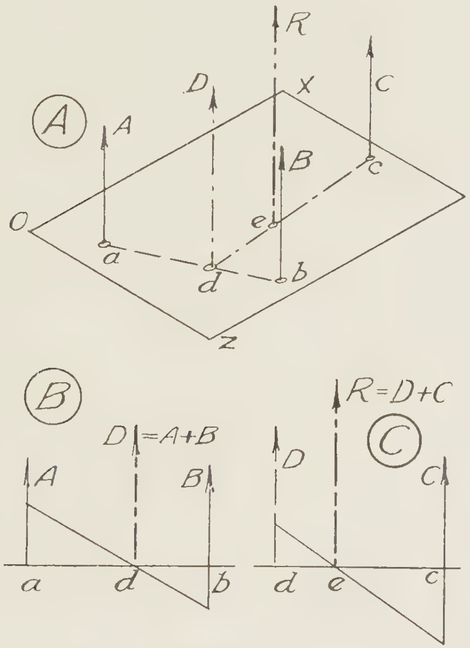


Fig. 49.

(Note. The graphical method of finding the points d and e is illustrated at (B) and (C) in Fig. 49.)

We have seen that the magnitude of R is equal to $C + D$, but force $D = A + B$; therefore the magnitude of R is equal to $A + B + C$. and R must be the required resultant of the three forces A, B and C, its position being fixed by the point e.

If the resultant is required in the case where there are more than three forces acting, it should be obvious that exactly the same method would apply, the forces being dealt with successively in pairs.

Similarly for the case where the resultant of a number of unlike forces is required, a single resultant could (except in special cases*) be found in the manner already described to replace this system of forces.

The **Equilibrant** of a number of like or unlike parallel forces which have a single resultant will evidently act in the same line as the resultant and be of equal magnitude but of opposite sense.

11. It follows therefore that, **for any system of parallel, non-coplanar forces which is in equilibrium, the algebraic sum of all the forces in a direction parallel to their lines of action will be equal to zero**, or: $\Sigma V = 0$, where V stands for all the forces acting in the general direction of the system of forces being considered. It follows from such a definition of V that $\Sigma H = 0$.

It is not difficult to show that $\Sigma V = 0$ and also $\Sigma H = 0$ must be true in all systems of parallel forces, even where the lines of action of the forces are inclined to the two planes of reference, V and H. For, since all the forces will be inclined equally to each of the planes, their components parallel to each of the planes will bear the same ratio to each other as in the case of the original forces, and their algebraic sums in each of these two directions must therefore still be zero. Hence $\Sigma V = 0$, and $\Sigma H = 0$, in each and every direction.

The following worked example will serve to make clear the method of applying these principles in the solution of actual problems.

Example. *Three vertical like forces* A, B *and* C *act at the corners of a horizontal triangle* abc *as shown in Fig.* 50. *To find the position and magnitude of the resultant of these three forces.*

The resultant D of A and B may be found either graphically or arithmetically. Arithmetically the work is as follows: Assume the force D $(= A + B)$ to be acting at point d and take moments about the point a. Then we have, since the moment of the resultant D must be equal to the moments of the forces which it replaces, see para. 13 (b).

$$D \times ad = B \times 10.$$

But $$D = A + B = 10 \text{ lbs.}$$

and $$B = 6 \text{ lbs.},$$

$$\therefore \ ad = \frac{6 \times 10}{10} = 6 \text{ ins.}$$

Similarly, having found the length of the line dc, either by calculation or by drawing, to be approximately 12·5 ins., then, taking moments

* Any system of non-concurrent forces may reduce to a couple, or to a couple and a single force. It is shown later, see para. 72, that in such a case the couple could only be balanced by another couple of equal magnitude but opposite sense.

about point d we have, since we know that R is equal to the sum of the three forces A, B and C,

$$R \times de = C \times dc, \quad \text{or} \quad de = \frac{15 \times 12{\cdot}5}{25} = 7\tfrac{1}{2} \text{ ins}$$

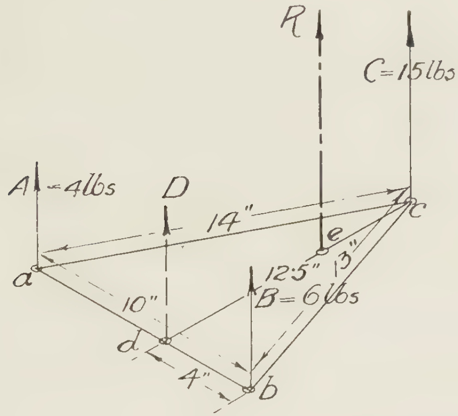


Fig. 50.

Experiment. The above work may be illustrated experimentally in the following manner. A flat panel, such as that shown in Fig. 51, is

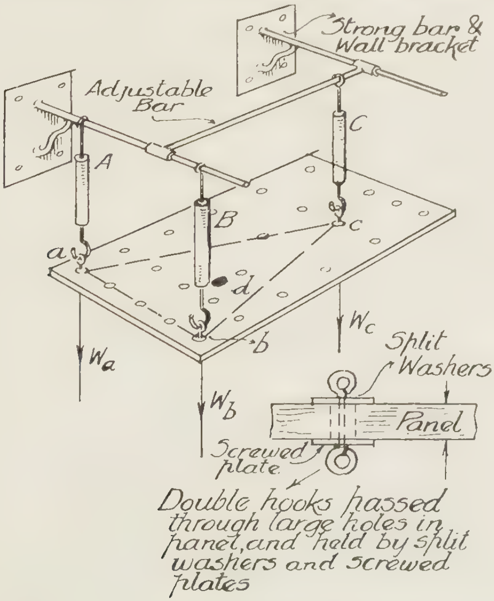


Fig. 51. Resultant of three non-coplanar forces.

suspended horizontally by means of three spring balances. The panel is so designed that the positions of the points of support a, b and c may be varied at will and, correspondingly, the points from which the balances are suspended. By means of the double hooks, which pass right through the panel as shown in Fig. 51, three weights of any suitable magnitude, W_a, W_b and W_c, are hung from the points a, b and c. *After adjusting the panel to a horizontal position* the readings of the balances are taken, the three weights are removed and a single weight R, of a magnitude equal to that of the three weights taken together, is placed upon the panel and moved from point to point until the three balances show the same readings as before.

The position (d) of the centre of the weight R having been marked the result so obtained should be checked by means of an arithmetical calculation for the given forces and distances.

43. To investigate the equilibrium of a system of parallel non-coplanar forces.

(Method 2.) *By the geometrical projection of the forces on to one or more planes.* Consider first the case of three parallel coplanar forces which are in equilibrium; see Fig. 52. Let these three forces A, B

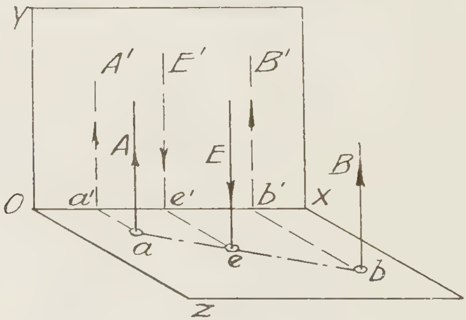


Fig. 52.

and E act *at right angles to the horizontal plane XOZ*, and let A', B' and E' be their projections upon the vertical plane XOY (which is not in this case either parallel to or at right angles to the horizontal line aeb).

Since the vertical plane XOY is parallel to the lines of action of the forces A, B and E, it is clear that, if we look upon their projections A', B' and E' as representing the components of the original forces in this direction, then these components, or, as we can conveniently call them, **Projected Forces**, will be equal in magnitude to the original forces, and

$$E' = A' + B', \quad \text{or } \Sigma V' = 0. \qquad \text{......(a)}$$

Also, since all the forces are at right angles to XOZ,

$$\Sigma H' = 0. \qquad \text{......(b)}$$

Further, since the projectors aa', bb' and ee' are parallel lines, then it follows geometrically that the distances $a'e'$ and $e'b'$ are in the same ratio to each other as the distances ae and eb. But this same ratio exists between the forces A and B, since

$$\frac{A}{B} = \frac{eb}{ae};$$

hence
$$\frac{A'}{B'} = \frac{e'b'}{a'e'},$$

or
$$A' \times a'e' = B' \times e'b', \text{ or briefly } \Sigma M' = 0. \quad \ldots\ldots(c)$$

In other words the system of projected forces A', B' and E' satisfies the conditions of equilibrium; see (a), (b) and (c). Similarly for any number of parallel forces not necessarily coplanar.

III. Hence it follows that, if **a projected system of forces is obtained—in the manner explained—from a system of parallel forces known to be in equilibrium, then the system of projected forces satisfies the conditions of equilibrium.**

Briefly this may be expressed as:

$$\Sigma V' = 0, \quad \Sigma H' = 0, \quad \Sigma M' = 0,$$

where V', H' and M' refer to the forces and moments of the projected systems.

(**Note.** It is not difficult to show that the above conditions are satisfied by the projected system no matter how the plane of projection is chosen, that is even when it is not parallel to the forces, but we shall omit any discussion of such cases here, since the simpler statement is sufficient for our purpose.)

Problems. In applying this method to the solution of problems, in which there are one or two unknown forces, it is usually necessary to obtain two projections of the system of forces, particularly if the position of one of the unknown forces is undefined, as only in this way can the position of the unknown force be found. It is then generally most convenient to take two "vertical" planes which are at right angles to each other. The following worked example will serve to make the method clear.

Example. *It is required to find the equilibrant of three unlike vertical forces,* A, B *and* C, *whose positions are indicated by the points* a, b *and* c, *as shown in plan in Fig.* 53, at which the lines of action of the forces penetrate the horizontal plane XOZ. *Forces* B *and* C *act away from the plane while the force* A *acts towards it.*

The plan is set out as shown in the figure and two elevations obtained about the lines OX and OZ. The distances of the projected forces from O, the point of origin, are then as indicated and, by applying $\Sigma V' = 0$ and $\Sigma M' = 0$, we may obtain the magnitude and position of E for each of the projected systems.

Consider the projected system in the plane XOY; E' is given by $9 + 7 - 5 = 11$ lbs., and clearly must act downwards.

Since we know that, for this projected system, $\Sigma M' = 0$, then, taking moments about point O, we have

$$E' \times x = (B' \times 5) + (C' \times 16) - (A' \times 2),$$

or

$$x = \frac{B' \times 5 + C' \times 16 - A' \times 2}{E'}$$

(compare with (i), para. 13).

Hence, inserting numerical values, we have

$$x = 13\tfrac{4}{11} \text{ ins.} \qquad \text{......(a)}$$

Dealing with the "projected system" on YOZ in a similar manner, we find

$$z = 8\tfrac{2}{11} \text{ ins.} \qquad \text{......(b)}$$

Using the two values (a) and (b) we obtain the point e, at which the equilibrant acts on the plane XOZ, by drawing two lines parallel to OX and OZ respectively; see Fig. 53.

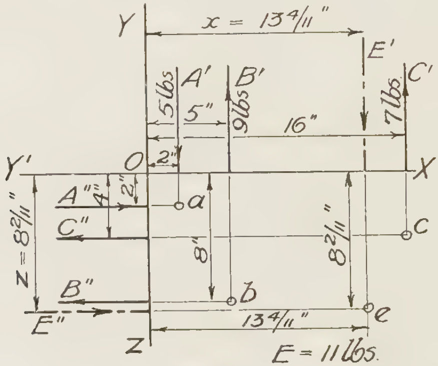


Fig. 53.

44. Experiment. *The experimental investigation of a system of parallel non-coplanar forces which is in equilibrium.*

The panel already described should be suspended as before from three points by spring balances; see Fig. 54. In this case, however, since we shall have to deal with forces which act both upwards and downwards, three balance weights should first be suspended from the three selected points a, b and c, these weights being of sufficient magnitude to prevent the panel tilting when the fourth weight W_d, representing the downward acting force D, is added to the panel.

The panel is first adjusted to the horizontal position, the readings of the balances being then taken as the "zero" readings in each case. The fourth force D is now added at some point d, and the panel once more adjusted to the horizontal. The *change* in the reading of each of the balances will indicate the forces acting at the respective points—a *decrease* in the reading indicating a *downward* acting force, and an *increase* in the reading indicating an *upward* acting force.

In the case illustrated the forces A and B will act upwards while the force C (as well as force D) will act downwards. The results so obtained should agree with those obtained by calculation for the same forces and distances.

It will be convenient at this stage to check the statement that the sum of the vertical forces is zero, or $\Sigma V = 0$.

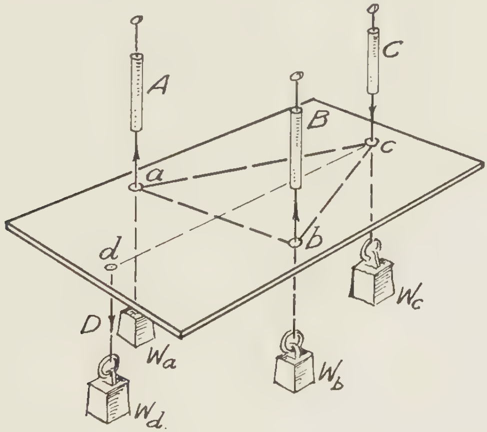


Fig. 54. Equilibrium of a system of unlike parallel non-coplanar forces.

45. To investigate the equilibrium of a system of parallel non-coplanar forces by considering the sum of the moments about a plane containing two of the forces. (Method 3.)

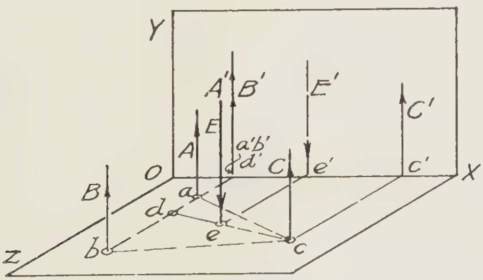


Fig. 55.

This method is really a special case of that described under Method 2, the projection being taken upon a plane which is at right angles to another plane containing two of the forces: see plane $BbaA$ in Fig. 55. In the "projected system" on plane XOY in Fig. 55 the projections of forces A and B will overlap;

hence, applying $\Sigma M' = 0$ to this "projected system", and taking moments about b', the moments of A' and B' will be zero, so that the sum of the moments of the remaining forces must also be zero, from which the value or position of the unknown force may be readily found.

The same results would have been obtained if we had taken the moments of the actual forces about an **axis** passing through and at right angles to two of them, say the axis represented by the line ab in the plane XOZ, and this is the method usually adopted in practice, see the Example worked below.

IV. Without further elaboration of the discussion we may in fact put this statement in a more general form and state that, **the algebraic sum of the moments of a system of non-coplanar forces (parallel or non-parallel) which is in equilibrium is zero about any axis, or $\Sigma M = 0$,** where the moments M are taken about any convenient axis. This must obviously be so, otherwise the system would rotate about some particular axis, which is contrary to the supposition of equilibrium.

Example. *A stand,* ABC *in Fig.* 56, *is formed of two pieces of wood fixed at right angles to each other. A weight* W *of* 24 *lbs. is placed at the point* E. *It is required to find the supporting forces which act at* A, B *and* C *when the stand rests on a level surface.*

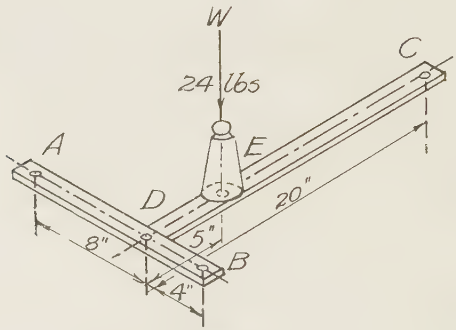


Fig. 56.

Taking moments about the plane containing A and B, i.e. about line AB as an axis, we have

$$C \times 20 = E \times 5, \quad \text{or } C = E \times \tfrac{5}{20} = 6 \text{ lbs. (upward).}$$

It follows from this that the effective upward force which must be applied at the point D to produce equilibrium must be equal to $(24 - 6)$ lbs., or 18 lbs., and this gives force D, which is of course the resultant of the two forces A and B.

Or $$A + B = 18 \text{ lbs.} \qquad \text{......(a)}$$

Taking moments about the plane containing the forces C and D, i.e. about the line CD, we have

$$A \times 8 = B \times 4,$$

whence $$8A = 4B$$

or $$A = B/2. \qquad \text{......(b)}$$

Substituting this value of A in the expression (a), we have:

$$B/2 + B = 18,$$

whence $$B = 12 \text{ lbs.,} \quad \text{and} \quad A = 6 \text{ lbs.}$$

46. The reactions of structures having more than two points of support. The following brief notes are intended to indicate the limits within which the methods just explained may be applied in practical cases.

(*a*) **Structures having three points of support.** The Example just completed will indicate the method of solution. Where more than one load has to be supported, then the loads may be readily reduced to a single load or resultant by the methods described above.

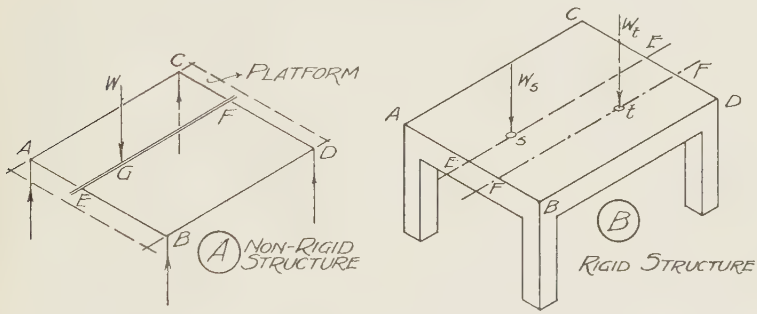


Fig. 57. Loaded structures with four points of support.

(*b*) **Structures having four points of support, the structure itself not being rigid.** Fig. 57 (A) illustrates a simple case that is of common occurrence. A load W is carried on a beam EF which in turn rests upon two other beams AB and CD. If, as assumed, the connections between the members are not rigid, then the supporting forces may be found by Method 1, that is by solution in successive planes. The method of solution will be the same even if the load be carried on a platform or non-rigid floor laid over the beams, as indicated by the dotted lines in Fig. 57 (A).

(*c*) **Rigid structures having four points of support (approximate solution).** The complete investigation of such cases would carry us beyond the limits set for this volume, since, to obtain an accurate solution, the strains taking place in the various parts of

the structure must be found. *Approximate* results may, however, be obtained as described below, and may be used when we desire to ascertain the reactions of a semi-rigid structure supported at four points, e.g. a travelling tower gantry.

Case I. *The load rests on an axis of symmetry*, see W_s on the line E–E in Fig. 57 (B). Then approximately the reactions at A and B (and also at C and D) will be equal to each other, and when added together will equal the reactions at the ends of the line EE, if EE be treated as a beam carrying the load W_s.

Case II. *The load is placed in any other position*, such as t in Fig. 57 (B). The reactions may be found *very approximately* by dealing with the case as though it were a non-rigid structure, the weight W_t being first borne by the imaginary "beam" FF. Alternatively we may assume that the load is distributed between the three *nearest* points of support (B, C and D in this case), as explained in the preceding Example.

Derrick Cranes

47. The Derrick Crane. See Fig. 58 (A). This type of crane—also known as the "stiff-legged derrick"—is one of the most useful employed by builders; it is readily moved, erected and dismantled, and does not require elaborate foundations or fixing. The base of the crane covers a triangular area, which usually forms either a right-angled or an equilateral triangle. The special feature of the derrick crane lies in the possibility of altering at will the inclination of the jib, and thus the radius at which the load may be swung. This operation is known as "derricking" or "luffing". The area over which a load can be raised or lowered *vertically* is thus very considerable; see shaded area in Fig. 58 (B).

In a properly designed derrick crane the alteration of the inclination of the jib can be carried out without appreciably altering the *level* at which the load is swinging. This is known as "level luffing". Such an arrangement can be shown to economise engine power during this operation and, in addition, it has some merits as a safety device when the load is being swung near the top of the jib, or when the load is being swung in over the cornice of a building. In the well-known Scotch derrick, level luffing is accomplished by the careful arrangement of the relative positions of the hoisting and derricking pulleys, together with the use of a special tapered derricking barrel, known as a "fusee" barrel. This gear is based upon that first invented by David Henderson of Renfrew in 1845.

The crane is prevented from overturning by "holding-down" loads at the extreme ends of the horizontal members, these loads

consisting of weights made up with bricks, stone, iron or other handy but heavy material. In the case of cranes which are permanently fixed the crane is usually secured to and supported upon three large blocks of concrete.

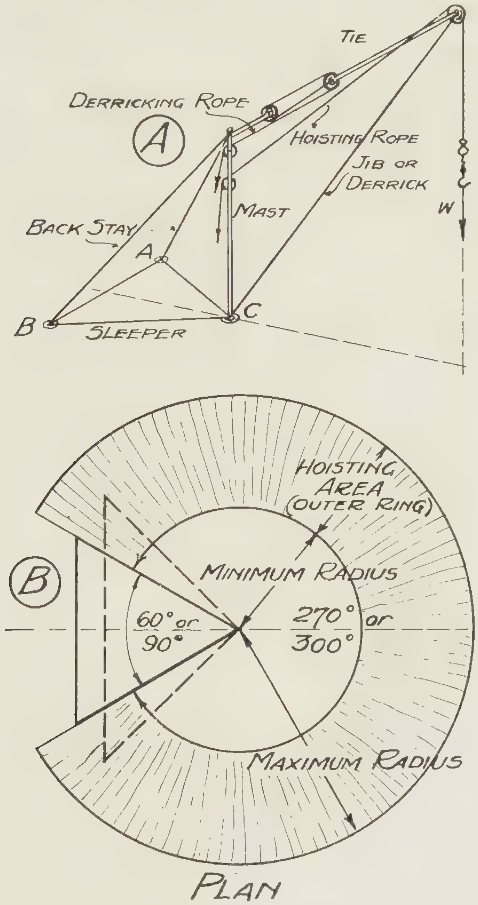


Fig. 58. The derrick crane.

The problem of ascertaining the magnitudes of the reactions at the points of support of a loaded derrick crane is readily solved by the methods just enumerated. In the absence of a ready-made rule which can be safely applied in all normal cases to give maximum "holding-down" loads, there is no question about the advantages to be gained from the possession of a fairly complete knowledge of the changes taking place in these reactions during

the raising and swinging of a load; this is particularly the case if, for some reason or other, it becomes necessary to fix the derrick in some unusual way, say to the framework of a building in progress.*

48. The reactions of a derrick crane. The example chosen is shown in plan and elevation in Fig. 59. As will be seen, the base of the crane forms an equilateral triangle of 30 ft. sides; the height of the vertical post or mast being also 30 ft., it follows that the vertical side frames form isosceles triangles, the two smaller angles being 45°. These relations are convenient for purposes of calculation, but it will be clear that the methods of solution employed below are also applicable where the relations differ from those set out above. The jib is 60 ft. long. The maximum radius at which the load can be swung is assumed to be 45 ft.; it is also assumed that the crane is capable of carrying a load of 5 tons at this radius.

The derrick frame being relatively light, its weight will be omitted from the calculations. Each joint in the frame will be assumed to be a pinned joint, though, for the present, we shall omit any consideration of the forces acting in the framework of the derrick and deal only with the external forces or reactions necessary to ensure the stability of the crane when carrying a load of 5 tons at the maximum radius. It may be noted here that when the points A and B are connected by an additional sleeper or in some other way, then the stationary part of the crane forms a triangulated and rigid frame in three dimensions.

The reactions. The derrick is supported at the three points A, B and C and it will be convenient to refer to the three supporting forces as forces A, B and C respectively. Since the only external load to which the crane is submitted is the load W at the end of the jib, and this is a vertical load, then it should be obvious from the work already done, that *vertical forces alone will be sufficient to ensure stability under load* W. This is an important point and should be clearly grasped by the reader before proceeding further: see small sketch in Fig. 59.

Stability Moment and Overturning Moment. If in the elevation given in Fig. 59 we consider moments about the point c', at the foot of the crane mast, then the force which is tending to overturn the crane is the weight W, and the **Overturning Moment** is the moment of W about c', that is in this case (5 tons × 45 ft.) or 225 ton-ft. The moment which resists this and which ensures the stability of the crane is equal to the moment of the combined forces A and B—generally due to holding-down bolts or weights

* The stability of a derrick crane erected upon a derrick tower is dealt with in Chap. VII.

placed at these points—about c'. This latter moment is called the **Stability Moment**, and in this case equals $(A + B) \times b'c'$.

Overturning moments and stability moments are sometimes referred to in the case of tall buildings, in which the overturning

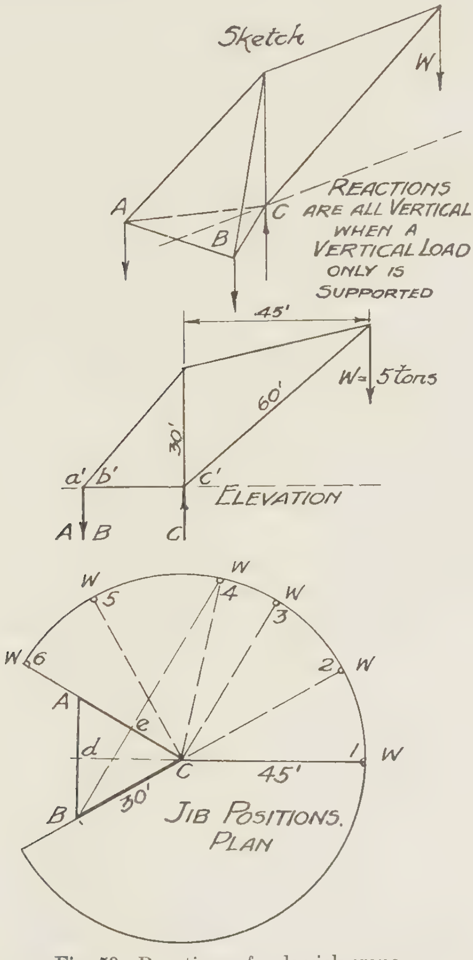


Fig. 59. Reactions of a derrick crane.

moment is generally due to wind pressure, while the magnitude of the stability moment depends upon the weight of the building and its width. In some American Building Regulations it is specified that the overturning moment shall not exceed 50 % (in some cases 75 %) of the stability moment.

Though the terms are not at present in general use in this country, it will be useful to retain them. They are easily understood and, though not easily applied to large buildings, we shall find them particularly valuable terms in dealing with the stability of scaffolding, etc.; see Problems at the end of this and the following chapter.

Positions of the jib. For convenience in dealing with the calculations six particular positions of the jib have been chosen (twelve, if both sides are considered); these positions are indicated in Fig. 59. Starting with the jib in the central position (Position 1), the jib is swung through an angle of 30° until its plan is in line with the side BC of the frame (Position 2). In Position 3 the jib lies at right angles to the side CA. In Position 4 the jib is so placed that the load W lies on a line drawn through e, the centre point of the line AC, to B. In Position 5 the jib is at right angles to the side BC, while in Position 6 it lies in line with the side CA.

Position 1. See Fig. 60. Both a plan and elevation have been drawn, but the plan will usually be sufficient as all the necessary dimensions can be obtained from it.

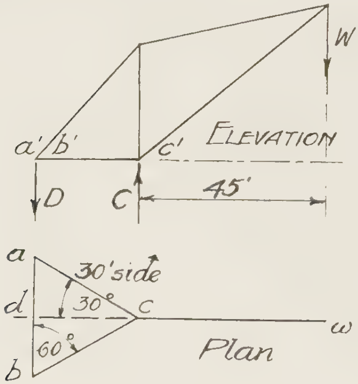


Fig. 60. Position 1.

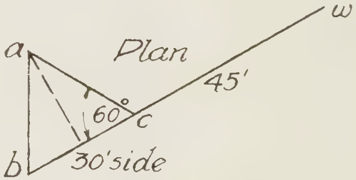


Fig. 61. Position 2.

Taking moments about the line dcw, it is evident that since a and b are equidistant from d, forces A and B must be equal in magnitude and both acting in the same direction. Also their resultant D must act at d,

or force $D = A + B$. D is a vertical force.

Consider the forces acting in the plane containing the forces C and W. Since wcd is a straight line and D is a vertical force, then D also acts in this plane.

Taking moments about point C, we have:

$$D \times dc = W \times wc. \qquad \text{......(a)}$$

But $$wc = 45 \text{ ft.}$$
and
$$\begin{aligned} dc &= ac \times \cos 30^\circ \\ &= 30 \text{ ft.} \times 0{\cdot}866 \\ &= 25{\cdot}98 \text{ ft.} \end{aligned}$$

Therefore from (a)
$$D \times 25{\cdot}98 = 5 \times 45,$$
$$\therefore D = 8{\cdot}66 \text{ tons, and obviously acts downwards.}$$

From this it follows that
$$A = B = 4{\cdot}33 \text{ tons, and both act downwards.}$$

C evidently acts upwards, but forces A, B and W all act downwards; and, since we know that $\Sigma V = 0$, we have:
$$\begin{aligned} C &= A + B + W \\ &= 4{\cdot}33 + 4{\cdot}33 + 5 \\ &= 13{\cdot}66 \text{ tons.} \end{aligned}$$

Position 2. See Fig. 61. If moments be taken about the line bcw it will be seen that there cannot be any force acting at a, since there is no other force acting off the line bcw to balance it, i.e. force $A = 0$. It is thus only necessary to consider the forces acting in the line bcw.

Taking moments about C we have:
$$B \times bc = W \times wc,$$
i.e.
$$B \times 30 = 5 \times 45,$$
whence
$$B = 7\tfrac{1}{2} \text{ tons, and acts downwards.}$$

The force C acts upwards and we have:
$$\begin{aligned} C &= B + W \\ &= 7\tfrac{1}{2} + 5 = 12\tfrac{1}{2} \text{ tons.} \end{aligned}$$

Position 3. See Fig. 62. The length of bg is found as in (a) and $= 25{\cdot}98$ ft.

Taking moments about ac we have:
$$B \times bg = W \times wc$$
or
$$B \times 25{\cdot}98 = 5 \times 45,$$
$$\therefore B = 8{\cdot}66 \text{ tons, and acts downwards.}$$

To find force A take moments about cw when:
$$A \times ac = B \times gc$$
or
$$A \times 30 = B \times 15,$$
whence
$$A = 4{\cdot}33 \text{ tons, and acts upwards.}$$

Without calculations consider the equilibrium of the forces about line ab, when it will be seen that force C must act upwards, and, since $\Sigma V = 0$, we have, calling downward acting forces negative,
$$A - B - W + C = 0,$$
that is
$$4{\cdot}33 - 8{\cdot}66 - 5 + C = 0,$$
$$\therefore C = 9{\cdot}33$$
or
$$C = 9{\cdot}33 \text{ tons, and acts upwards}$$

Position 4. See Fig. 63. The length of wg is obtained as follows:

$$\begin{aligned} gw &= \sqrt{wc^2 - gc^2} \\ &= \sqrt{2025 - 225} \\ &= \sqrt{1800} \\ &= 42{\cdot}4 \text{ ft.} \end{aligned}$$

Taking moments about ac we have:

$$B \times gb = W \times wg$$

or $$B \times 25{\cdot}98 = 5 \times 42{\cdot}4,$$

whence $$B = 8{\cdot}16 \text{ tons, and acts downwards.}$$

To find forces A and C first take moments about bgw. Evidently forces A and C have equal magnitudes and act in the same direction; also their resultant (G) will act at g and force $G = A + C$.

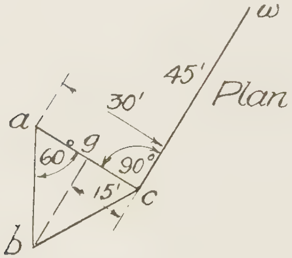


Fig. 62. Position 3.

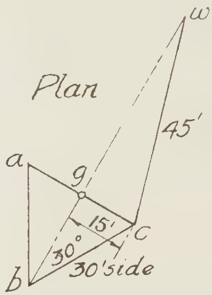


Fig. 63. Position 4.

Now consider the forces G, B and W acting in the line bgw and taking moments about b we have:

$$G \times bg = W \times bw$$

or $$G \times 25{\cdot}98 = 5 \times (25{\cdot}98 + 42{\cdot}4),$$

whence $$G = 13{\cdot}16 \text{ tons, and acts upwards.}$$

$$\therefore \; A = C = \tfrac{1}{2}G = 6{\cdot}58 \text{ tons, both acting upwards.}$$

Position 5. See Fig. 64. Taking moments about bc we have:

$$A \times ah = W \times wc$$

or $$A \times 25{\cdot}98 = 5 \times 45,$$

whence $$A = 8{\cdot}66 \text{ tons, and acts upwards.}$$

Taking moments about wc we have:

$$B \times bc = A \times hc$$

or $$B \times 30 = A \times 15,$$

whence $$B = 4{\cdot}33 \text{ tons, and acts downwards.}$$

To find force C. Since $\Sigma V = 0$, we have:

$$A - B - W + C = 0$$

or $$8{\cdot}66 - 4{\cdot}33 - 5 + C = 0,$$

whence $$C = 0{\cdot}67 \text{ ton, and acts upwards.}$$

Position 6. See Fig. 65. Consider moments about line *caw*, obviously force B has zero magnitude.

Consider equilibrium of forces in line *caw*, take moments about a, then

$$C \times 30 = W \times 15,$$

whence $$C = 2\tfrac{1}{2} \text{ tons, downwards.}$$

Since $\Sigma V = 0$ we have $A = W + C = 7\frac{1}{2}$ tons, and must act upwards.

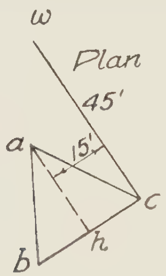


Fig. 64. Position 5.

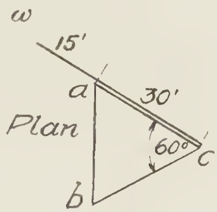


Fig. 65. Position 6.

The forces obtained in the above investigation may be tabulated in a convenient form as shown below.

Stability forces in a derrick crane

No.	Position of jib angle from (1)	Forces (tons)		
		A	B	C
1	0°	− 4·33	4·33	13·66
2	30°	0	− 7·5	12·5
3	60°	4·33	− 8·66	9·33
4	70·5° approx.	6·58	- 8·16	6·58
5	120°	8·66	− 4·33	0·67
6	150°	7·5	0	− 2·5

Note. A negative sign indicates that the force acts downwards and that a holding-down force or load of this magnitude is necessary for stability.

These reactions or forces, which are here called "stability forces", have been plotted in the form of a graph in Fig. 66. The forces will of course be repeated when the jib is swung in a clockwise direction from Position 1, the graphs being symmetrical if all the positions are taken.

The positions giving maximum upward and downward stability forces should be noted carefully as well as the corresponding magnitudes of the forces called into play. For A these occur near Positions 5 and 3, respectively, and for B near Positions 5, and 3 respectively. For C the maximum upward acting force occurs in Position 1.

It should be noted that all the reactions falling below the line represent downward acting forces and therefore necessitate the use of suitable holding-down loads, chains or bolts. In particular it should be noted that a downward acting force is called into play at C as the jib nears the Position 6 (and of course $6_{,}$); if the weight of the engine or motor—usually fixed at or near to C—is not

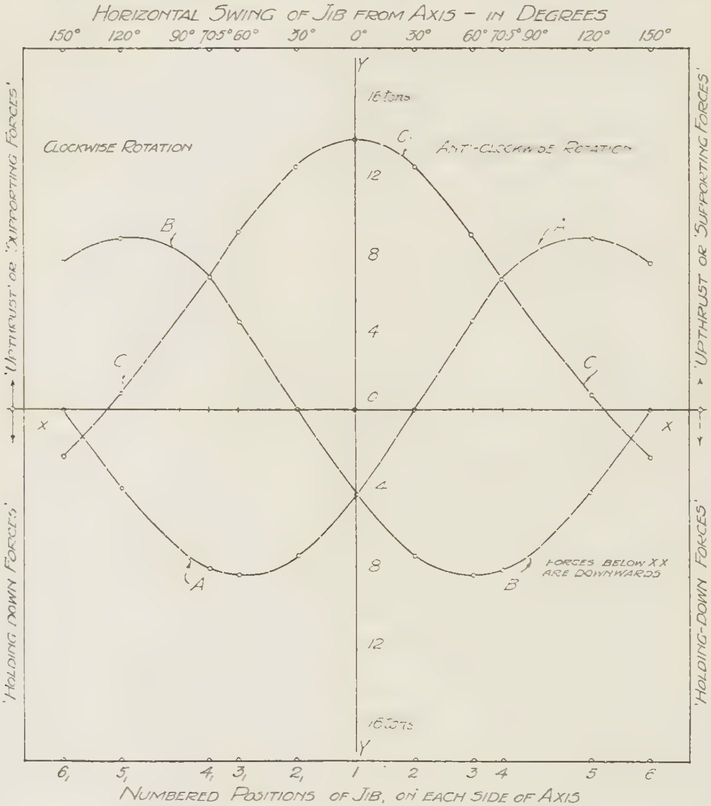


Fig. 66. Stability forces in a loaded derrick crane.

sufficient to counterbalance this force, then C must be adequately weighted or otherwise secured.

Other results would of course be obtained for different proportions and loading, hence each case should be separately investigated. *For safety the holding-down loads should be at least twice the magnitude found from such calculations in any particular case.*

49. Experiment. The experimental investigation of the cases dealt with above is readily carried out by means of the apparatus used in

the experiment described in para. 42. The panel is suspended from three balances as shown in Fig. 67, but at C, which is placed some distance back from the edge of the board, the suspending chain passes through a rod CW, which is free to rotate about C. CW represents the jib in plan while A, B and C are the supporting forces. As these supporting forces may be acting either up or down, according to the position of the jib, the following method of measuring the forces should be adopted.

Weights are suspended below the panel from each of the points A, B and C. These weights should be greater in each case than the largest downward acting force likely to be called into play at any one of these points. After having noted the zero readings the weight W is hung

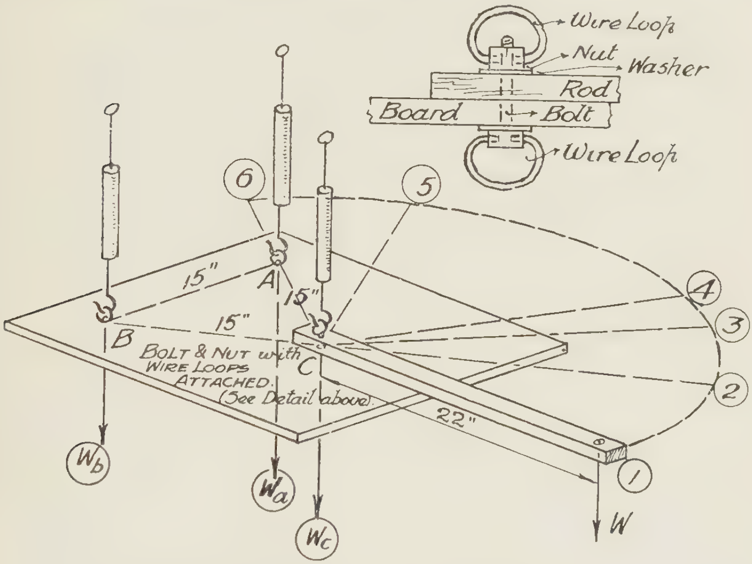


Fig. 67. Experiment—Reactions of a derrick crane.

from the jib which is moved so as to take up successively the positions already indicated, the panel being adjusted to the horizontal each time. The *change in reading* of each of the balances is then noted. If the zero reading is increased, then a corresponding upward force is necessary for stability. If the zero reading is reduced, then a downward acting force of a magnitude equal to this reduction is necessary for stability.

An alternative method for finding the reactions is to use a model of a derrick crane, say constructed on a scale of 1/12 or 1/24, the reactions being found in the manner explained in para. 63 in the next chapter.

Problems VI

1. If weights of 8, 10 and 12 lbs. are suspended from the corners of an equilateral triangle of 30 in. sides, find the magnitude and position of the balancing force (E).

2. If in Prob. 1 the 8 lbs. force is reversed in direction, find the position and magnitude of the balancing force (E).

3. In Fig. A is shown in diagrammatic form a type of hoisting appliance sometimes used during the carrying out of repairs to an existing building. A long girder EG is attached to two floor beams within the building—or two beams are specially provided. The motor-driven hoisting block at G can be traversed along the girder and from this block the load W is suspended. If E and F divide each of the floor beams into two equal parts, and EF is 20 ft., while FG is 8 ft., find the maximum supporting forces at A, B, C and D when a load of 1 ton is to be lifted and traversed from G to E on the girder.

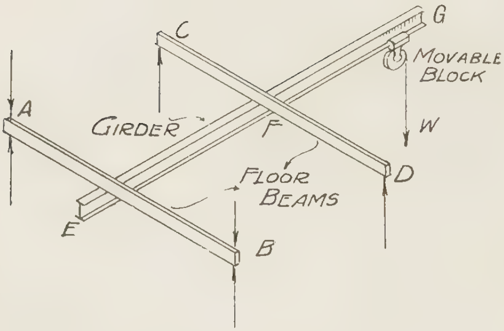


Fig. A. Hoisting plant.

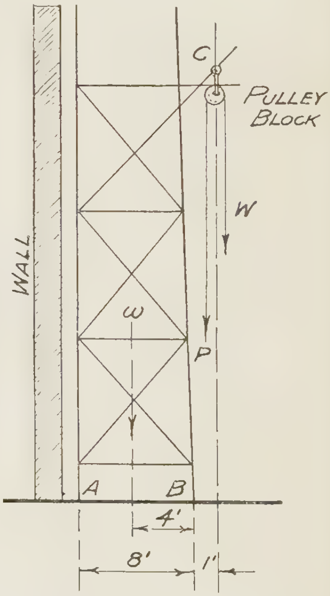


Fig. B. Hoisting tackle on a scaffold.

4. Fig. B shows a scaffold erected against a wall, against which it is assumed merely to rest. A pulley block at C overhangs the base of the scaffold by 1 ft., and the total weight of the scaffold (w), which may be taken to act on a vertical line through the centre of the base AB, is 2000 lbs. If the pull (P) on the hoisting rope is equal to W, the weight to be lifted, find the maximum weight which may be lifted so that the overturning moment does not exceed 50 % of the stability moment. (Note. Take moments about B.)

5. Find the reactions for a derrick crane in Position 1 and Position 6 (see Fig. 59), when W is 4 tons, the hoisting radius is 60 ft. and the sides of the base of the crane are each 45 ft.

CHAPTER VII

THE EQUILIBRIUM OF SYSTEMS OF CONCURRENT NON-COPLANAR FORCES. THE FORCES ACTING IN MASTS AND IN DERRICK TOWERS

The analysis of simple systems of concurrent forces which are in equilibrium, but which do not act in one plane (non-coplanar), may be conducted on lines very similar to those adopted in dealing with systems of non-coplanar parallel forces. The work involves relatively simple principles which are set out in the next two paragraphs.

50. The minimum number of concurrent forces necessary to keep a particle in equilibrium.

(*a*) **Parallel forces.** Imagine the particle P in space to be acted upon by a number of parallel forces such as forces A and B; see Fig. 68. From what we have already done it should be clear that *two such forces will be sufficient to keep* P *in a state of equilibrium* provided that the two forces A and B are equal in magnitude and act in opposite directions. To be concurrent they must of necessity act in the same straight line, i.e. they are coplanar.

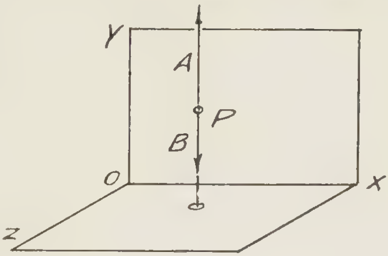


Fig. 68.

(*b*) **Concurrent inclined forces.** Let a third force C inclined to the other two be applied at the point P. The two original forces cannot now act in the same straight line and the lines of the forces must all be inclined to each other, as shown in Fig. 69. Now we know that three concurrent forces are sufficient to keep a particle in equilibrium, but can we be sure that these three forces must of necessity act in the same plane? It would appear on investigation that this must be so: for, consider it to be otherwise, and imagine that the force C is inclined to the plane containing the two forces A

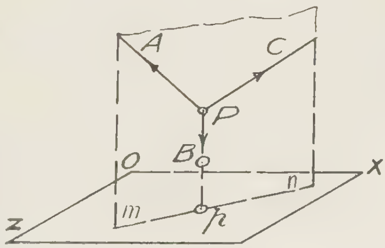


Fig. 69.

and B: then, under such conditions the force C would have a component acting at right angles to the plane of A and B. Then, for the equilibrium of the point P, it would appear that there should be another force, equal in magnitude but opposite in direction, to balance this component of C. But forces A and B act wholly within the plane containing them and therefore cannot supply such a balancing component. As there is no other force acting at P it follows that, for equilibrium, the force C must lie in the plane containing the forces A and B, in other words *the three forces must be coplanar.* From this we see that *three concurrent inclined forces are sufficient to keep a particle in equilibrium provided they are coplanar.*

Experiment. The truth of the last statement may be simply demonstrated as follows, see Fig. 69. A weight is suspended by a cord from the small ring P and the ring is supported by two other cords inclined to each other and to the vertical cord. A horizontal board is then placed beneath the weight and the positions of the cords projected on to this surface (any other plane at right angles to one of the forces may of course be selected); the projection of the forces is given by the lines pm and pn and the point p. The lines pm and pn will be found to lie in one and the same straight line, which will pass through p, thus demonstrating geometrically that the three forces are coplanar.

(*c*) Concurrent non-coplanar forces. We have just seen that it would be impossible to maintain a particle in equilibrium by the application of only three inclined and concurrent forces unless those forces acted in one and the same plane. Hence it would appear that *a system of concurrent non-coplanar forces cannot be in equilibrium with less than four forces.*

Let us consider this case in detail. Suppose a fourth force D is added to the three forces A, B and C of the last sub-paragraph (*b*); see Fig. 70. If we take the plane containing any two of the forces, say A and C, then both the other forces must be inclined to this plane; for, suppose it were not so, and that the three forces A, B and C were all coplanar while the force D was not, then D would have a component at right angles to the plane containing the other three forces. But, obviously, none of the other three forces could have a component at right angles to the plane since they act in that plane, hence there would be no force to balance this component of D, that is the particle P could not be in equilibrium. It follows that,

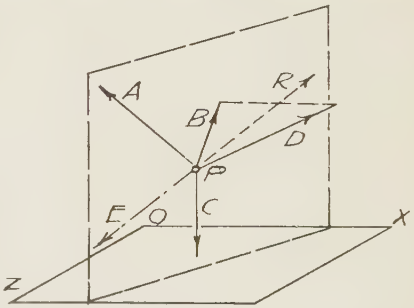


Fig. 70.

for the equilibrium of P the forces B and D must be inclined to the plane containing the remaining forces A and C and *vice versa*. This argument further shows that, *four non-coplanar concurrent forces are sufficient to keep a particle in equilibrium.*

(The experimental treatment of four non-coplanar forces is dealt with in para. 55.)

51. The analysis of systems of concurrent non-coplanar forces which are in equilibrium. In systems of concurrent non-coplanar forces which are known to be in equilibrium but in which all the forces are not fully defined, it is possible to obtain a solution by replacing the system by an equivalent system of forces which is simpler—usually coplanar—and which contains fewer unknown quantities.

Equivalent systems of forces. *Two systems of forces may be said to be equivalent if either system may be substituted for the other without changing the state of rest or of motion of the body upon which they act.* (Obviously, therefore, systems which have the same resultant or are in equilibrium are equivalent systems and are interchangeable.)

To reduce a system of non-coplanar forces to one consisting only of coplanar forces. In the system of four non-coplanar forces represented in Fig. 70 let the forces B and D be replaced by their resultant R, then:

(*a*) **The force R lies in the plane of the forces B and D**; for, imagine R to be reversed so that it becomes the equilibrant (E) of the two forces B and D, then since these three forces, B, D and E, are in equilibrium they must be coplanar, that is E, and therefore R, lies in the plane containing B and D; and

(*b*) **The force R also lies in the plane of the forces A and C.** The effect of the new force R upon the particle P would be exactly the same as that of the two forces which it replaces, i.e. the particle P would still be in equilibrium. But if the particle P is kept in equilibrium by the three forces A, C and R, then these three forces must be coplanar, i.e. the force R also acts in the plane containing A and C.

From (*a*) and (*b*) it is evident that *the force* R *must lie along the intersection of the two planes*, the plane containing the forces B and D, and the plane containing the forces A and C.

This is an important statement which is of great value in solving problems dealing with non-coplanar forces. In the above case it has been possible by its aid to replace one pair of forces by a third force acting in the plane of the remaining pair of forces. Hence, in any system containing more than four non-coplanar forces which are in equilibrium, it would be similarly possible to reduce the

system, step by step, to a system containing only three coplanar forces.

We may now put this and the preceding statements concerning systems of non-coplanar forces into a general form upon which to base our later work.

I. If a system of three concurrent forces be in equilibrium, then these three forces must act in one and the same plane.

II. Since a system of three inclined concurrent forces is sufficient to maintain a particle in equilibrium, then it may replace any other system of concurrent forces—not necessarily coplanar—containing a larger number of forces which is likewise capable of producing equilibrium.

III. If, in a system of concurrent non-coplanar forces which is in equilibrium, we consider the replacement of two of the forces by a third, acting in the plane of two (or more) of the forces already given, then the third force must lie along the intersection of the two planes, the plane containing the two forces to be replaced, and the plane containing the given forces.

IV. If, in a system of concurrent non-coplanar forces which is in equilibrium. a plane be taken which contains two of the forces, then all the other forces may be replaced by a single force acting in that plane and which balances the two selected forces. This single force must evidently be the resultant of all the forces which it replaces.

Problems. The problems to be dealt with in this chapter have been grouped according to the difficulty of the geometrical work involved in their solution. The first group can be dealt with by means of simple sketches and direct calculations, while the second group, in which the methods are of more general application, involve fairly difficult geometrical work.

52. Problems dealing with systems of non-coplanar forces which can be solved by calculation alone. In these examples the forces bear some simple relation to each other or are symmetrically grouped about the known force or forces.

Example 1. *A freely pivoted rod, projecting horizontally from a vertical surface, is supported by two equally inclined stays as shown in Fig. 71. A load of 1000 lbs. is suspended from the outer end of the rod. It is required to find the magnitudes of the forces acting in the rod and in the stays.*

If a vertical plane be taken which contains the force W and the force acting in the rod OC, it is clear from the figure that this plane will also

contain the line OD, which bisects the triangle OAB formed by the two stays. Further, from the preceding discussions, para. 51 (IV), it should be clear that the resultant (say "force D") of the two forces A and B, which must act along this line OD, must also be the equilibrant of the two forces W and C.

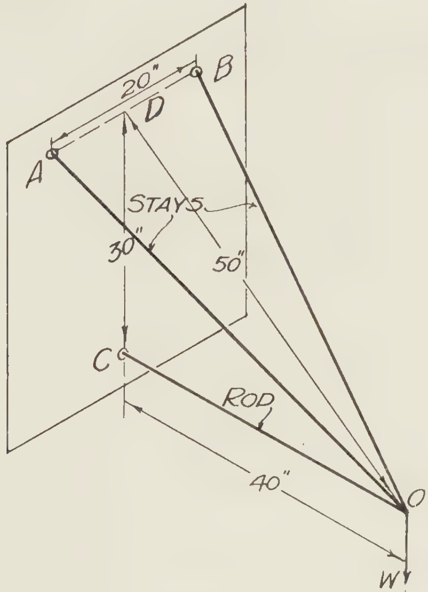


Fig. 71.

Consider the equilibrium of the point O under the action of the three forces W, C and D; see Fig. 72. From the frame diagram a force triangle has been drawn from which it will be seen that: (a) since the force triangle

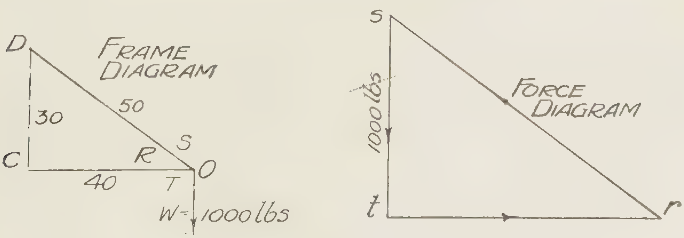


Fig. 72.

has the same shape as the frame diagram, then the ratio of the sides will be the same; and (b) the vertical force, which is represented by the line st, has a magnitude of 1000 lbs.

Hence the force in the rod OC, represented by tr,

$$= \frac{1000 \times 40}{30} = 1333 \text{ lbs.};$$

the force diagram shows this to be a compressive force.

Also the force acting in line OD

$$= \frac{1000 \times 50}{30} = 1666 \text{ lbs. (tension).}$$

Consider now the forces acting in the two stays OA and OB. The figure AOB being symmetrical it follows that these forces have equal magnitudes. The shape of the triangle AOB is set out in Fig. 73 and the line of action of the force OD marked upon it. Let the length OD represent to some scale the magnitude of this force. Draw De parallel to BO and Df parallel to AO. Then evidently $DeOf$ is the force parallelogram for the force OD. Join ef cutting DO in g. It follows from the dimensions and from the symmetry of the figure that $eg = 5$ ins. and $Og = 25$ ins.

Hence $\qquad eO = \sqrt{5^2 + 25^2} = \sqrt{650} = 25{\cdot}5$ ins. nearly.

But $$\frac{\text{force in } OA}{\text{force in } OD} = \frac{Oe}{OD} = \frac{25{\cdot}5}{50},$$

Hence
$$\begin{aligned}\text{force in } OA &= \text{force in } OB \\ &= \frac{\text{force in } OD \times 25{\cdot}5}{50} \\ &= \frac{1666 \times 25{\cdot}5}{50} \\ &= 850 \text{ lbs. approx.}\end{aligned}$$

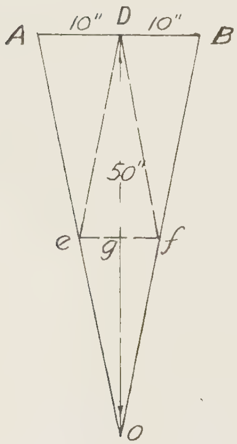


Fig. 73.

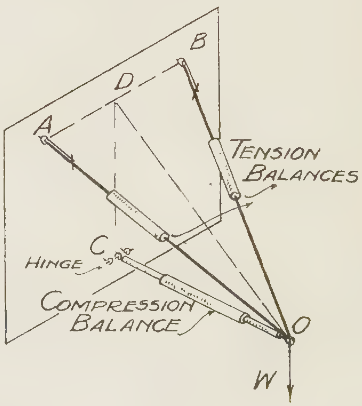


Fig. 74.

Experiment. The above case can be illustrated experimentally and other cases investigated by arranging a piece of apparatus as described below.

A light rod, having a compression balance incorporated in its length, is freely hinged to a vertical board as shown in Fig. 74. The two stays

include tension balances and also some simple means of adjusting their length. Initial readings of the balances should be taken before the weight W is added. After the weight has been added the lengths of the stays and rod should be adjusted to the original length and the increases in the readings of the balances noted.

One of the stays may now be removed and the position of the other stay altered until it takes up the position midway between the positions occupied originally by the two stays. After having adjusted the length of this stay so that the rod OC is again horizontal, the magnitude of the force acting in this single stay should be found to be equal to the resultant of the forces which acted in the two stays and the magnitude of the force in the rod should be unaltered.

Example 2. *A pair of sheer legs* (sometimes known as an *A*-frame) *is to be used for lifting a weight of 2 tons. The dimensions of the frame are as shown in Fig. 75. A single rope is used to maintain the frame in the position shown.* (A rope in the opposite direction should be attached to P for safety.) *Find the forces acting in the legs of the frame and in the rope.*

Obviously the legs are in compression and the rope in tension. The legs being symmetrically disposed about the weight W, the forces in them will be of equal magnitude.

If w be the point immediately below W, and points C and w be joined, this line will cut the line AB at the point D so that the length of AD is equal to the length DB (10 ft.). If PD be joined, it should be clear that the line PD marks the intersection of the plane containing the forces in PC and PW with the plane containing the forces in PA and PB. Along this line PD will act the force PD which, as the resultant of forces A and B, may replace them, when it will act as the equilibrant of forces W and PC. We will consider the latter case first. The figure $CPwD$ has been redrawn in Fig. 76, and from it has been set out the force triangle srt for the forces acting at P. By extending sr to w and drawing wt at right angles to sw it will be seen that this figure is similar to that composing the frame diagram, hence the ratio of the sides will be the same.

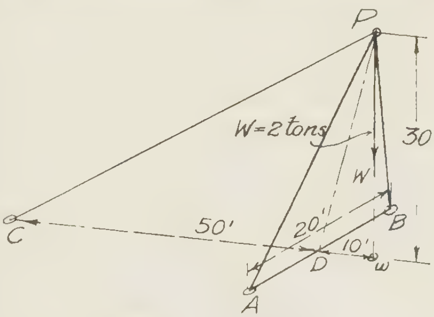


Fig. 75. Sheer legs.

Consider the frame diagram.

$$\text{The length of } CP = \sqrt{60^2 + 30^2} = \sqrt{4500}$$
$$= 67 \text{ ft. approx.}$$
$$\text{The length of } PD = \sqrt{30^2 + 10^2} = \sqrt{1000}$$
$$= 31{\cdot}6 \text{ ft. approx.}$$

The ratios of the sides of the force triangle may now be marked as shown in small circles in Fig. 76. The magnitude of the force W,

represented by the side rs, being known, the other forces may be found as follows.

The force acting in the rope and represented by tr

$$= \frac{2 \times 31{\cdot}6}{50}$$
$$= 1{\cdot}264 \text{ tons.}$$

The force acting in PD and represented by st

$$= \text{resultant of force in } PA \text{ and } PB$$
$$= \frac{2 \times 67}{50}$$
$$= 2{\cdot}68 \text{ tons.}$$

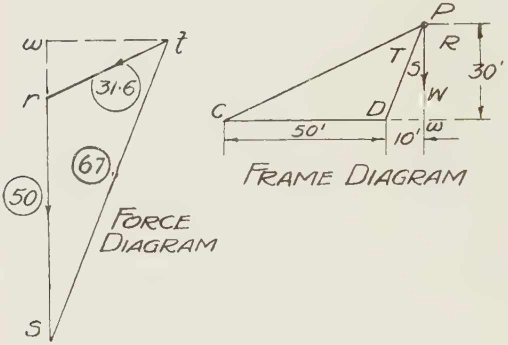


Fig. 76.

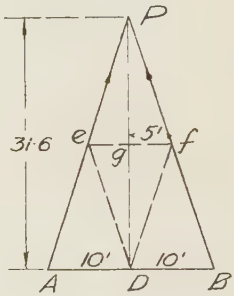


Fig. 77.

Consider next the forces acting in the plane of the legs; see Fig. 77. If to some scale the line PD is made to represent the force acting in PD, then we may complete the force parallelogram $PeDf$. Then, as before, we have:

$$gf = \frac{DB}{2} = 5 \text{ ft.}$$

and $$Pg = \frac{31{\cdot}6}{2} = 15{\cdot}8.$$

Hence $$Pf = \sqrt{15{\cdot}8^2 + 5^2} = 16{\cdot}6 \text{ ft. approx.}$$

The forces represented by the sides of the parallelogram being in like proportion, we may obtain the magnitudes of the forces as follows:

The compressive force acting in PA

$$= \text{the compressive force acting in } PB$$
$$= \frac{PD}{2} \times \frac{16{\cdot}6}{15{\cdot}8} = \frac{2{\cdot}68}{2} \times \frac{16{\cdot}6}{15{\cdot}8}$$
$$= 1{\cdot}41 \text{ tons approx.}$$

53. Problems dealing with systems of non-coplanar concurrent forces which necessitate the use of graphical methods.

General Case. For the sake of simplicity we will consider a system consisting only of three forces in equilibrium. We already know that such a system cannot be non-coplanar, but we also

know that such a system may replace a larger non-coplanar system. At present it is therefore sufficient to realise that principles which are demonstrated to be true for a system of three concurrent forces acting in space must also be true for any larger but equivalent system of non-coplanar forces.

Let the particle P in Fig. 78 be kept in a state of equilibrium by the action of three forces A, B and C. For convenience the plane XOZ has been chosen so that the line of action of the force C is at right angles to it. The actual lengths of PA, PB and PC are taken to represent, to some suitable scale, the magnitudes of the three forces. The projections of these forces on the vertical plane XOY are given by $p'a'$, $p'b'$ and $p'c'$ respectively, and on the horizontal plane XOZ by pa, pb and pc.

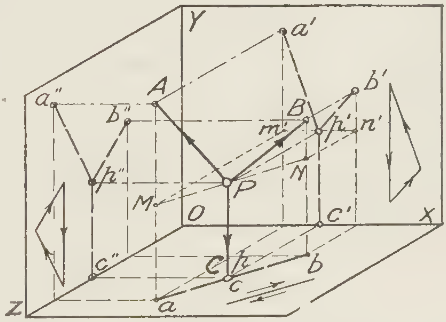


Fig. 78.

As we have already seen, since the force C is vertical and the forces are coplanar, the projection of the system of forces in the plane XOZ will be the straight line apb. Consider the equilibrium of the point P in a horizontal direction. Since C is a vertical force it has no horizontal component. We are thus left with the horizontal components of the forces A and B. Through P draw the horizontal line MPN, the point M being on the projector drawn through A to a, and the point N likewise on the projector joining B to b. Then obviously PM and PN represent the horizontal components of the forces A and B respectively. Since there are no other horizontal forces to be considered, then for equilibrium the force PM must equal the force PN and these forces must act in the same straight line. It follows that the projections of PM and PN, i.e., pa and pb, will lie in the same straight line (apb), and will be of equal length. In this case $apbc$ represents what we shall call the "projected system of forces" in the horizontal plane XOZ, the projection of the force PC being represented by the point c.

Thus we are able to state that, for the "projected system" in the plane XOZ, the condition $\Sigma H = 0$ is satisfied and, since there are no forces acting at right angles to apb, we have that $\Sigma V = 0$ is also satisfied. (The condition $\Sigma M = 0$ must obviously be satisfied since the forces act on a particle and keep it in equilibrium.) The "projected system" of forces in the plane of XOZ therefore satisfies the conditions of equilibrium for coplanar forces and it should be possible to draw a closed force polygon (in this case a straight line); see Fig. 78.

(a) Consider next the "projected system" of forces in the vertical plane XOY. Since the lines PM and PN are horizontal, they will project into a horizontal line $m'p'n'$ on the vertical plane and, as before, and by the rules of projection, $p'm'$ will be equal to $p'n'$. As these represent the horizontal components of the two forces $p'a'$ and $p'b'$, and $p'c'$ is vertical, evidently the condition $\Sigma H = 0$ is satisfied by this "projected system".

(*b*) Again the lines MA and NB represent in magnitude and direction the vertical components of the forces A and B respectively in the original system, and these vertical components must evidently be together equal to the other vertical force C to produce equilibrium in a vertical direction. All these lines being vertical they are unaltered in length by being projected upon the vertical plane XOY, hence we are able to state that in the "projected system" the vertical forces are likewise in equilibrium and that the condition $\Sigma V - 0$ is thus satisfied by the "projected system".

Thus it follows from (*a*) and (*b*) that it must be possible to draw a closed force polygon (see para. 6) for the "projected system" of forces shown on the vertical plane XOY, see Fig. 78, and similarly for the system projected on to the plane YOZ *or any other plane.*

Without elaborating this argument further it will be sufficient to state that it is likewise true for any larger system of concurrent forces (not necessarily coplanar) which is in equilibrium; so that, *for any system of concurrent forces—coplanar or non-coplanar—in equilibrium a closed polygon can be drawn for any "projected system" obtained in the manner explained above.* By this means we can solve problems dealing with non-coplanar forces of the type set out below.

It is important to remember that the above statement involves the following more general statement: *that for any system of concurrent forces—coplanar or non-coplanar—which is equilibrium,*

$$\Sigma H = 0,$$
$$\Sigma V = 0$$
and
$$\Sigma M = 0.$$

54. Example. *A load of* W *tons is supported by a set of sheer legs, as shown in Fig.* 79. *It is required to find the magnitude and character of the forces acting in each leg when the legs are of unequal lengths.*

This problem is typical of those which have to be solved in practice. It will be noticed that of the four forces acting at the point P only one, the weight W, is fully known. It should be quite obvious that the forces acting in the legs will all be compressive forces and no more need be said on this point.

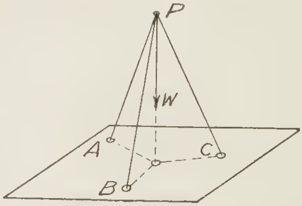


Fig. 79.

If a plan and elevation be drawn for this case as in Fig. 80, it will be seen that we cannot complete the force polygon in either the vertical plane or the horizontal plane for the "projected systems" since we have insufficient data. Thus in the elevation we have four forces acting at the point o' of which only one is fully defined. In the plan only the directions of the forces are known, the force W being represented by a point.

To surmount this difficulty we apply the principle which was given in para. 51 (IV).

The necessary steps in the solution of the problem may be summarised as follows:

Using Fig. 80:

(*a*) One of the planes of projection is selected so as to be parallel to the plane containing two of the forces, one of which is completely defined. In the present case a vertical plane is chosen which is parallel to the plane containing the force W and the leg OC.

(*b*) The other plane of projection should if possible be chosen so as to be at right angles to one of the two selected forces. W is at right angles to the horizontal plane in this case. It follows from these conditions that the plan of the plane containing forces W and C is the straight line *oc*; see Fig. 80.

(*c*) If the unknown forces in A and B be replaced by their resultant say D, acting in the plane of the two forces W and C, we know that these three forces W, C and D will be in equilibrium and, if we can ascertain the direction of the force D, we can then complete the force triangle.

Since the force D is to be the resultant of forces A and B, it must act in the same plane as forces A and B. Similarly we know that it must act in the plane containing the forces W and C. Hence it follows that the force D must act along the line OD in which these two planes intersect. The horizontal trace of the plane containing forces W, C and D

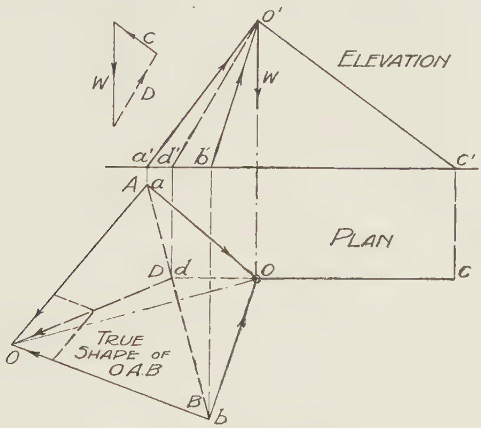


Fig. 80. Sheer legs—graphical solution.

is given by the line *cod*. Similarly the horizontal trace of the plane containing the forces A, B and D is given by the line *adb*. These lines intersect in the point d which must therefore lie on the intersection of the two planes. The point O is common to both planes and we are thus able to complete the plan (*od*) and elevation ($o'd'$) of the line OD in which the force D acts.

We are now able to complete the force triangle for the forces W, C and D as projected in the vertical plane, see Fig. 80; from this we find the true magnitude of the force in the leg OC and of the force D.

Several methods are now available for the completion of the problem. The following is usually most easy to apply:

(*d*) If the true shape of the triangle OAB is found and the position of the line OD marked upon it, see plan in Fig. 80, then, by means of a parallelogram of forces, it is possible to find the components of force D along the lines OA and OB; these give the magnitudes of the forces A and B.

55. Experiment. The foregoing problem may be illustrated experimentally by arranging a piece of apparatus as shown in Fig. 81. By using this arrangement in which all the forces are tensile forces, the experimental work is simplified. The force D need not necessarily be vertical. The magnitude of the force acting in each direction is measured by a spring balance as shown.

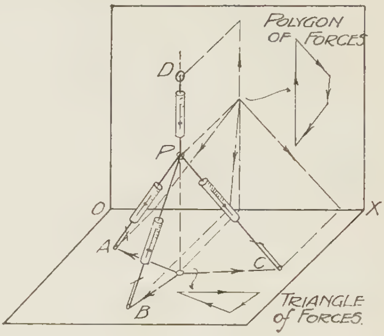


Fig. 81. Experiment—system of four non-coplanar forces.

Having ascertained the forces acting in each of the cords and also the true positions (by projection or otherwise) of the points A, B, C, D and P, the cords are removed. From the particulars obtained the projected systems are drawn, the force polygons completed and the magnitudes of the forces obtained in the manner already described. The values thus obtained should be compared with those obtained experimentally.

We may now proceed to apply and expand the principles and methods explained above in the treatment of several groups of important practical problems, which arise in connection with the use of stayed masts, derrick cranes, derrick towers, etc.

Stayed Masts

56. Assumptions. In order to keep the work at this stage within reasonable limits of difficulty, the following assumptions will be made; their full significance will be appreciated as the work proceeds:

(i) That all poles, masts, "towers", or built-up members, are secured at each end by connections which, within the limits of movement expected of them, allow these members to move as in a universal joint.

(ii) That all stays or guys, unless otherwise stated, are capable of resisting tensile stresses only; if submitted to compressive forces they will thus go out of action (but see later note on "Initial Tension", para. 60).

Experiment. *Vertical mast with horizontal stays.* Let a light rod, PO, be freely pivoted to a horizontal board as shown in Fig. 82, the rod being maintained in a vertical position by three *horizontal stays* A, B and C. All the stays should be adjustable in length; this may be easily achieved with sufficient accuracy for our present purpose by using at the end of each stay a length of brass chain attached to a hook, the stay may then be lengthened or shortened by "letting-out" or "taking-in" as much chain as is necessary. The mast should likewise be adjustable in length; the use of some form of double-acting turnbuckle is probably the readiest means of making this adjustment. In order to measure the forces acting

in each of the members the stays should include light tension balances while the mast should include a compression balance. The experiment should proceed as follows:

(*a*) Having adjusted the stays until *PO* is vertical read all the balances and tabulate the values so obtained;

(*b*) By altering the lengths of the stays change the pull acting in each of them, being careful, however, to adjust these until *PO* is once more vertical; read the balances once more and tabulate the values;

(*c*) An additional series of readings may be obtained after one or two more additional stays have been attached to the mast.

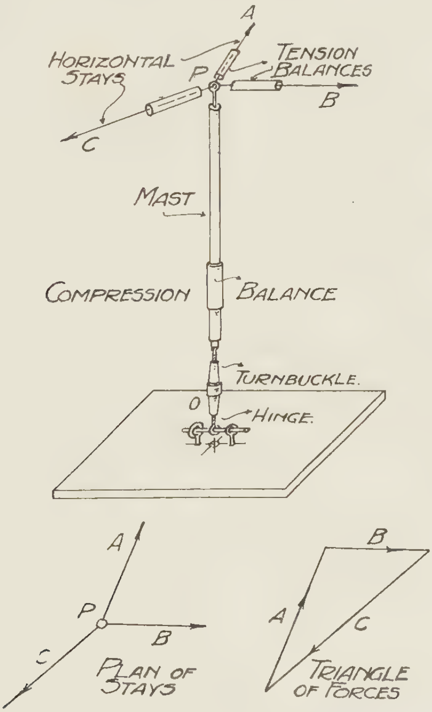


Fig. 82. Mast with three horizontal stays.

The following points should now be noted:

If the three forces acting at *P* in case (*a*) and case (*b*) be set out, as shown in plan in Fig. 82, then it should be possible to draw a closed force triangle for each set of forces, thus indicating that in each case they constitute a system of three coplanar forces in equilibrium. Similarly in case (*c*) it should be possible to draw a closed polygon, where more than three forces act in the stays attached to *P*.

If one of the three horizontal forces acting at P in the first two cases be known, then it will be possible to complete the force triangle and to ascertain the magnitude of the remaining forces. In the third case, if four forces act at P, then it will be necessary to know two of the forces (three in the case of five forces, etc.) before the force diagram can be completed.

The reading of the balance in the mast will not be altered in case (*b*) after the forces have been changed, showing that, apart from the vertical force due to the weight of the apparatus (the mast, stays and balances), the applied horizontal forces do not induce any forces in the vertical mast.

57. Experiment. *Vertical mast with one inclined stay.* The same apparatus may be used as in the last experiment but, if possible, there should be a fixed vertical rod, as at MN, to which the end of the stay B may be secured; it is then possible to alter the inclination of B while keeping it in the same vertical plane; see Fig. 83. The experiment should proceed as follows:

(*a*) Adjust the stays A, B and C—all being horizontal—until PO is vertical and read the balances as before.

(*b*) Lower the outer end of the stay B, as already described, until the stay makes an angle of say about 30° with the horizontal. Alter the pull in this stay until PO is once more vertical and the stays A and C occupy the same positions as before. (The length of PO will also have to be adjusted to bring the stays A and C to a horizontal position.)

Read all the balances and tabulate the forces as before.

(*c*) Plot the force acting in the stay B to a suitable scale as shown in Fig. 83, and from it obtain its vertical component (B_v) and its horizontal component (B_h). Also draw in plan, see Fig. 83, the three horizontal forces acting at P in the first case (*a*).

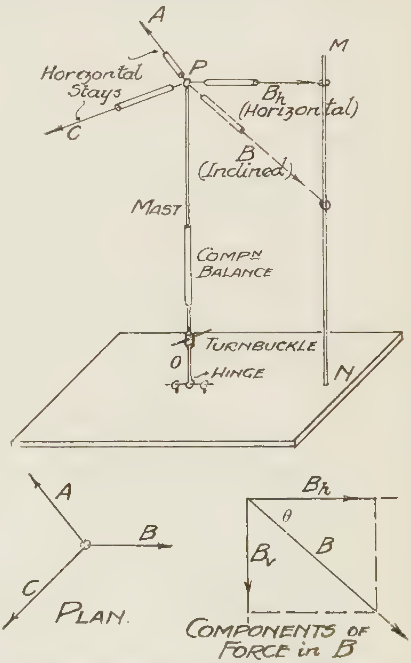


Fig. 83. Mast with one inclined stay.

The following points should be noted:

The forces in the stays A and C are the same in both cases.

The horizontal component of the force B is equal to the force which acted in this stay when in a horizontal position; and (i) this

force (B_h) together with the two forces A and C constitute a system of three coplanar forces in equilibrium; (ii) the force acting in the mast was increased in the second case by an amount equal to the vertical component of B, viz. B_v. (iii) If θ be the inclination of the stay PB to the horizontal, then the horizontal component $B_h = B \cos \theta$ and the vertical component $B_v = B \sin \theta$ (see Problems VII, 1).

58. Experiment. *Vertical mast with three inclined stays.* Using the same apparatus arrange three stays so that, while they are all inclined to the horizontal, they maintain the mast PO in a vertical position as before; see Fig. 84.

Ascertain the forces acting in the stays and mast. (When ascertaining the force acting in the mast allowance must be made for the weight of the apparatus. This can be done with sufficient accuracy for our purpose by reading the balance ("zero reading") when the stays are hanging loosely from the mast.)

Having obtained the various dimensions, set out each of the forces and from them obtain their vertical and horizontal components.

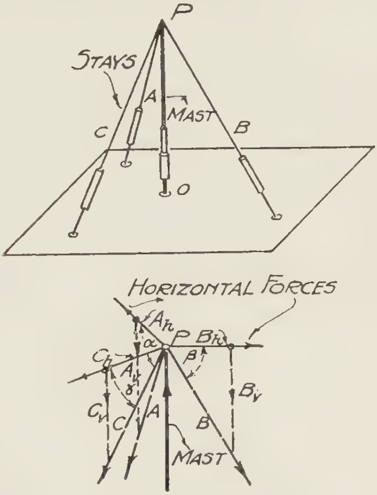


Fig. 84. Mast with three inclined stays.

The following statements should now be tested:

V. When a mast is held in position by a series of stays secured as already described, then **the total force acting in the mast is equal to the sum of the vertical components of the forces acting in the stays.**

In this case if P be the total compressive force in the mast due to the forces in the stays, then $P = A_v + B_v + C_v$, or, see (ii) and (iii) in para. 57,

$$P = A \sin \alpha + B \sin \beta + C \sin \gamma;$$

see Fig. 84. (If one of the inclined forces acts *upwards from the horizontal plane passing through the top of the mast*, then the vertical component of that force may be expressed as a negative quantity in the above expression, since it will *reduce* the total compression in the mast.)

VI. When a mast is secured in a vertical position by a series of inclined stays as here described, then **the horizontal components of the forces acting in the stays form a balanced system of concurrent forces acting at the top of the mast**; see Fig. 84 and (i) para. 57.

(If the mast is not vertical, then the components should be taken at right angles and parallel to the mast respectively.)

Without further elaboration it may be pointed out that the above statement is of quite general application and true no matter how many stays are used or in what direction they act provided, of course, that they all act at the same point.

In the case where a mast has more than one set of stays, see Fig. 85, other factors arise which we do not propose to deal with at this stage; for example it would be possible, by the faulty adjustment of the lower set of stays to bend the mast at B. If such were done it should now be clear to the reader that the horizontal components of the forces acting in the inclined stays attached at B could not be in equilibrium, so that some additional force was necessary to produce equilibrium and was evidently supplied by the bent mast. In practice, the unnecessary stressing of the mast should be avoided by careful adjustment of the stays.

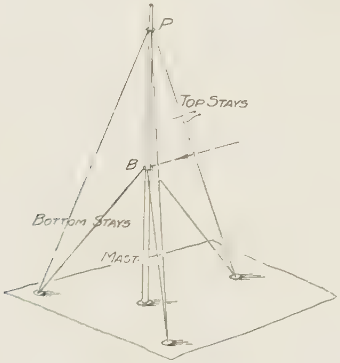


Fig. 85.

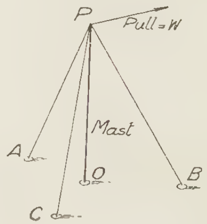


Fig. 86.

59. Stayed masts subjected to a varying pull. If a stayed mast be used for the purpose of resisting a steady and permanent pull (W), horizontal or otherwise, see Fig. 86, then the rope or cable by which that pull is transmitted may be looked upon as an additional stay. The problem of ascertaining the forces acting in the stays in such a case can therefore be dealt with on the lines described in the preceding paragraph.

The same remarks would apply to a case where the pull applied to the mast varied in amount but not in direction; only two cases need usually be treated, viz. when the pull has its maximum value and also when it has its minimum value.

When, however, a mast such as that shown in Fig. 86 is subjected to a pull W which may vary both in amount and direction, then the

problem requires further consideration. It should be clear that in such a case the forces acting in each of the stays will pass through a series of values according to the direction and magnitude of W. Even where there are only three stays supporting the mast the complete investigation of such a case is both lengthy and difficult. Fortunately for our purposes there are practical considerations which enable us to reduce the necessary work, particularly in the case of the somewhat elementary investigations which we are here undertaking. Thus, except where local conditions make it impossible, the stays are usually arranged symmetrically about the mast and a minimum number are employed so long as the requirements of safety are met.

Again it should be clear, from a consideration of the system of horizontal components acting at the top of such a mast, that a system of three stays is all that is necessary to maintain equilibrium; the plan of such a system is given in Fig. 87. If one of the three stays were to fail the mast would collapse for certain positions of the pull W. It is therefore usual to have four or more stays in each set. (It should be noted that, from a safety point of view, four symmetrical stays are no better than three, since failure of one stay may similarly cause the collapse of the mast.)

At this stage we will limit our investigations to the two cases of three or four stays symmetrically arranged in each set; it should thus be only necessary for us to investigate fully the forces acting in one stay in each case.

Case I. *Three stays symmetrically arranged*; see Fig. 87. Produce the lines AP and CP to a and c respectively. Now it should be obvious that so long as the force W acts within the angle cPa the stay B will tend to go out of action—since it cannot resist a compressive force—and need not therefore be considered (but see note on "Initial Tension", para. 60). We are thus left with the forces acting in the two inclined stays, the force in the mast and the force W. (This case is important and is dealt with in para. 61.)

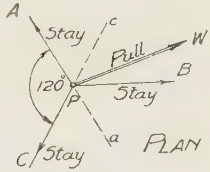


Fig. 87.

Let us next consider the conditions which exist when the line of action of the force W coincides with say the line Pc in Fig. 87. It seems clear that in this case the stay PC would take the whole of the force W since, if it were not so, it would mean that the stays A and B were in compression; this by our assumption we know to be inadmissible. We can in fact go further and complete our statement by saying that *when the pull* W *and one of the stays lie in the same vertical plane, but on opposite sides of the mast, then the force in that stay is at a maximum.*

Case II. *A mast supported by four stays symmetrically arranged,* as shown in plan in Fig. 88. It will be clear, from similar considerations to those set out above, that when the force W acts between any two of the stays, say C and B, those two stays will tend to go out of action and the pull will be resisted by the two stays D and A on the opposite side of the mast; the case thus reduces to that mentioned in Case I.

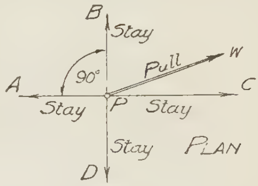


Fig. 88.

Similarly, when the line of action of W coincides with the general direction of one of the stays, say B, then the opposite stay will take the whole of this pull and the force acting in it will be at a maximum. In this case it is clear that the stays A and C could not contribute any components to the forces acting along BPD, since they are acting at right angles to that line.

In all those cases where the total pull is taken by one stay it is of practical importance to note that, although in theory these two opposite forces (really three including the force in the mast) are sufficient for equilibrium, the remaining stay or stays are necessary to maintain the stability of the mast which, without them, would tend to fall to one side or the other.

From the above considerations—and from others which need not be set out here—we see that, except in very complex cases, *the majority of problems dealing with stayed masts subjected to an applied force can be resolved into that of a mast with two inclined stays subjected to a given pull.* Because of its general importance this case is dealt with at greater length in para. 61. Before doing this, however, an important point must first be considered, viz. that of initial tension.

60. Initial tension in stays. In order to prevent a mast swaying unduly on the application of relatively slight forces, such as pressure due to the wind, it is always necessary to "pull the stays tight". In other words by straining the stays after they have been placed in position initial stresses or tensile forces are applied through the stays to the mast. It will be obvious that any stay which is subjected to initial tension will have to take any other tensile stress which may be put upon it as an addition to the tension already existing. If, however, the stress put upon a flexible stay acts in the opposite direction, then it will reduce the force due to initial tension until, finally, the stress in the stay becomes zero, or, more strictly speaking, until the stress acting in the stay is merely that due to its own weight when hanging loosely.

The maximum stress in a stay will therefore be the sum of the

initial stress and the maximum tensile stress engendered in the stay by outside forces. In building work, where stayed masts are used mainly for temporary purposes, the initial stresses are not usually considerable in comparison with the maximum forces which the stays may be called upon to resist. To simplify the solutions we will—unless otherwise stated in a problem—ignore the effect of initial tension and retain the first assumption, viz. that a flexible stay goes out of action when it is submitted to a compressive force. Provided that initial tension is allowed for when calculating the size of stay to be used, this assumption will not lead to serious error and will, in fact, be on the safe side.

61. Mast, with two inclined stays, subjected to a varying pull. If a piece of apparatus be set up similar to that shown in Fig. 89 (A), in which we have a vertical mast, two inclined stays and a horizontal pull W symmetrically placed, then it will be observed that this case is the same as that dealt with in para. 52 and illustrated in Fig. 71.

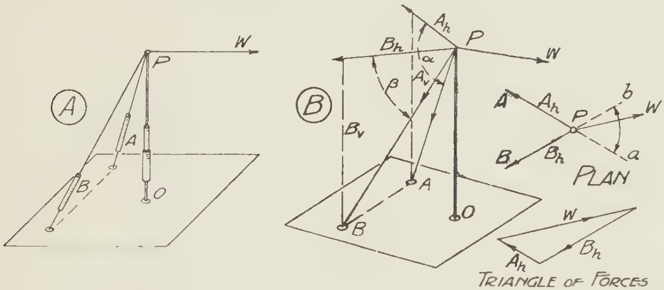


Fig. 89. Mast with two inclined stays.

It is therefore only necessary at this stage to deal with the case where the force W is not symmetrically placed. It must be remembered that, in the case where the mast is held upright by only two flexible stays, the line of action of W must not pass outside the angle aPb shown in the plan Fig. 89 (B).

A complete solution of this case may be obtained by means of the geometrical methods already described in para. 53, but where it is desired to investigate several positions the following methods, which utilise the results of the preceding paras. 57 and 58, will generally be found to be both easier and quicker.

(*a*) See Fig. 89 (B). Consider the forces acting at P in the horizontal plane containing force W. Let α and β be the angles at which the stays A and B are respectively inclined to the horizontal. Then the three horizontal forces acting at P and maintaining it in equilibrium in this

plane are A_h, B_h and the force W. The magnitude and direction of W are already known, and we are therefore able to complete the force diagram as shown in Fig. 89 (B). From this the values of A_h and B_h can be obtained. Then, from (iii), para. 57,

$$\text{the force acting in the stay } A = \frac{A_h}{\cos \alpha},$$

and

$$\text{the force acting in the stay } B = \frac{B_h}{\cos \beta}.$$

(*b*) Since the pull W acts horizontally it contributes no component to the total force acting in the mast, this latter force must therefore be equal to the sum of the vertical components of the forces acting in the two inclined stays; therefore we have, from (iii), para. 57,

$$\text{force acting in mast} = A_v + B_v,$$

or

$$= (\text{force in } A) \sin \alpha + (\text{force in } B) \sin \beta.$$

Experiment. Using the apparatus indicated in Fig. 89 (A) the above case may now be dealt with experimentally, and the results so obtained compared with those obtained by calculation. At least two cases within the angle aPb, shown in Fig. 89 (B), should be taken. In addition a special case may be taken in which the pull W is inclined in an upward direction from P. (From what we already know about non-coplanar forces it should be clear that if the force W rises above the plane containing the two stays A and B, then the mast will be subjected to tension instead of compression.)

The case finally considered may be one in which the force W acts in the plane containing one of the back stays and the mast. Experimentally it will be found difficult to maintain the mast in equilibrium in this case, but it should be clear that in this case the single stay supplies the horizontal component which balances the force W.

62. Experiment. *To ascertain the forces acting in a simple stayed-mast derrick crane.*

This is a useful application of the work already done and will serve to illustrate the methods to be employed in dealing with practical problems.

The apparatus which is necessary in this case is shown in Fig. 90. The mast should preferably be solid so as to reduce the necessary adjustments. To it is attached a long jib and a short tie in both of which balances may be included if desired. For the sake of simplicity the stays should be horizontal and it will not be necessary to use more than three.

The experiment should proceed as follows:

(*a*) Before attaching the load W, adjust the lengths of the stays until PO is vertical; then read all the balances. These readings should be treated as "zero readings".

(*b*) Apply the load W and adjust the stays once more until PO is vertical (a simple way is to move the lower pivot O, after having adjusted the stays until they make the same angles with each other as before). Also adjust the triangle made by the mast, the jib and the tie to its original shape. The *increase* shown by the balances in each case over the "zero readings", will indicate the magnitudes of the forces acting in the members and produced by the application of the load W.

(*c*) By means of a force triangle, see Fig. 90, obtain the forces acting in the tie and in the jib. (These should correspond to the values obtained experimentally.)

(*d*) By the same method obtain the magnitudes of the vertical and horizontal components of the force in the tie; see Fig. 90.

(*e*) Ascertain by graphical methods the equilibrant of the three forces acting in the stays. This should be equal in magnitude and correspond in direction to that of the horizontal component of the force in the tie.

(*f*) The jib should now be swung to other positions and similar investigations made for each position.

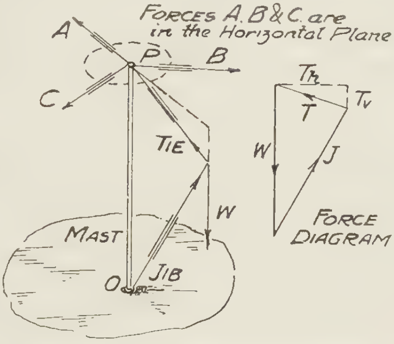


Fig. 90. Stayed-mast derrick crane.

The following points are important and should be noted; they may be checked experimentally if not already included in the experiment above:

(i) *The horizontal component of the force in the tie is constant for the same load and inclination of the jib.* Having once found this component, the problem of finding the forces acting in the horizontal stays (or the horizontal components of the forces in the stays if the latter are inclined) may therefore be dealt with by the methods described in the preceding para. 61.

(ii) *The force in the mast is similarly constant and equal to the vertical component of the force in the tie.* This force is increased if inclined stays take the place of the horizontal stays.

(iii) *The force acting in the jib is also constant for the same load and inclination.* It contributes nothing to the force in the mast.

Derrick Cranes

63. Experiment. *To ascertain the forces acting in the framework of a derrick crane.*

For this experiment a model derrick crane should be made up as indicated in Fig. 91. As in the last experiment balances may be included in the jib and tie, but the mast and horizontal members should be solid. It is an advantage, however, if the model is so constructed that any of these solid members may be replaced by members containing balances and *vice versa*. If possible one or both of the back stays should include balances which are capable of reading both tension and compression. All the joints of the frame should be pin-joints. To render the crane stable, even when loaded, the base ABC should be secured to three blocks as shown in Fig. 91. The weight of each of these blocks should exceed the greatest upward acting force likely to be called into play at any one of the points of support.

It may be noted here that in certain positions of the load the back stays will be subjected to compression; flexible stays would therefore be unsuitable in these positions.

The experiment should proceed as follows:

(*a*) Place the jib in "position 1"; see Fig. 91. Adjust the back stays until the mast is vertical and take the "zero readings".

(**Note.** All the members containing balances should be adjustable in length.)

(*b*) Add the load *W* and readjust the lengths until the various members take up their original positions. Read all the balances and tabulate the increases.

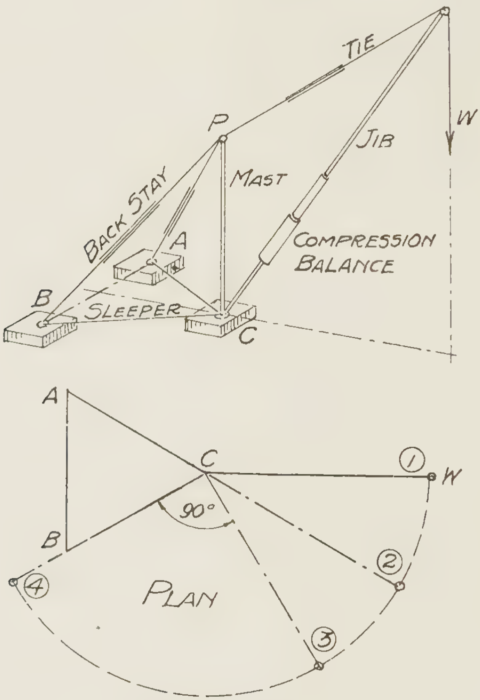


Fig. 91. Experimental derrick crane.

(**Note.** The procedure of (*a*) and (*b*) must be repeated for each of the cases mentioned below, but if *W* remains constant throughout the experiment it will not be necessary to take any further readings in the jib or tie.)

(*c*) The jib should now be swung into other positions and the forces acting in the various members ascertained for each position and tabulated.

For a very complete investigation at least the six positions mentioned in para. 48 should be tested. For our present purpose the four positions indicated in Fig. 91 should suffice.

(*d*) So as to be in a position to check the results fully the reactions at A, B and C should be obtained. This may be done by supporting one of the blocks at A, B or C on the platform of a small weighing machine, the other two blocks being raised until the base of the crane is level. The force acting at one of the points of support can then be obtained for each position of the jib, after which the values for one of the other points of support may likewise be obtained. (If two weighing machines are available the results can be obtained more speedily.) Since the sum total of the reactions must be equal to W (allowing for the weight of the crane), the reactions at the third point can be obtained by calculation.

In order to allow for the weight of the crane and blocks the following procedure should be adopted. The reading for one (or two) of the points of support is first obtained without the load W. This will be the "zero reading". The load W should then be added and a new reading obtained. If the first reading has been *increased*, then the increase will give the magnitude of the *upward acting force* which is necessary to support the crane at that point. If the reading on the weighing machine has *decreased*, then it will give the magnitude of the *downward acting force* which must be applied at that point to maintain the crane in position.

(*e*) Having obtained and recorded the magnitudes of all the forces for each position mentioned above, the cases should be analysed graphically—or otherwise—and the experimental results compared with those obtained by analysis.

The reactions may be calculated by the methods described in para. 48.

The graphical analysis contains no serious difficulty but, since the problem is one of considerable practical interest, it is dealt with in some detail in the next paragraph. By the intelligent use of the principles enunciated in the preceding paragraphs it is possible to reduce to reasonable dimensions the work involved in obtaining a complete solution.

64. Outline of the analysis of the forces acting in the framework of a derrick crane. The same four positions will be taken as were mentioned in the preceding paragraph. For convenience these have been indicated again in Fig. 92. The following important points should be noted, since they lead to certain reductions in the work and afford means of checking the results as the work proceeds:

(i) The pull in the tie remains constant for the same load and inclination of the jib. The horizontal component of this force may be obtained as shown in Fig. 92, and is represented on the force diagram by T_h. The horizontal force T_h, together with the horizontal components of the forces acting in the back stays, form a system of three forces in equilibrium for every position of the jib. Knowing this we may, by the methods described in para. 61, obtain the forces acting in the back stays. The alternative method is to follow the geometrical procedure described in para. 53, which usually takes much more time.

(ii) The force in the jib will likewise remain constant. The vertical component of this force, see J_v in the force diagram in Fig. 92, will not contribute anything to the force in the mast, but will contribute a constant component to the total force to be balanced by the reaction at C.

(iii) The horizontal component of the force acting in the jib, see J_h in Fig. 92, will be balanced by the forces acting in the two side sleepers; for any position of the jib these three forces will therefore form a system of three forces in equilibrium.

(iv) The force acting in the mast will be made up of the vertical components of the forces acting in the tie and the two back stays. The former force will be constant for the same load W but the latter will vary as the jib takes up the several positions.

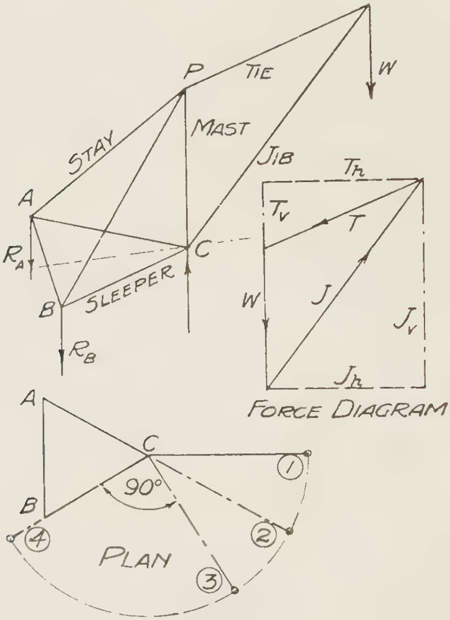


Fig. 92. Forces in the framework of a derrick crane.

(v) The total force in the mast together with the vertical component of the force in the jib, see (ii), will be equal in magnitude to the reaction at C.

(vi) The reactions at A and B will be equal to the vertical component of the force acting in the corresponding back stay.

(vii) So long as the crane is only subjected to vertical forces no force will act in the back sleeper. Its use, however, is necessary to keep the frame from collapsing. (In practice if the crane and stability weights rest on the ground the use of a back sleeper is of course unnecessary.)

Notes on the special conditions arising in each position.

Position 1. The jib being placed symmetrically this is not a difficult case. The following method may be adopted in preference to a purely graphical solution:

(*a*) See Fig. 93. Find by drawing the horizontal and vertical components of the forces in the tie and the jib respectively.

(*b*) Find the horizontal components of the forces acting in the back stays A and B. These components (A_h and B_h) form a system of coplanar forces in equilibrium when taken with the horizontal component (T_h) of the force acting in the tie. Having found the horizontal components find the actual forces acting in the back stays by calculation, using the methods described in para. 61.

(*c*) The magnitudes of the reactions at A and B will be equal to the vertical component of the force acting in the stay in each case, i.e., see Fig. 93, $R_A = A_v$ and $R_B = B_v$.

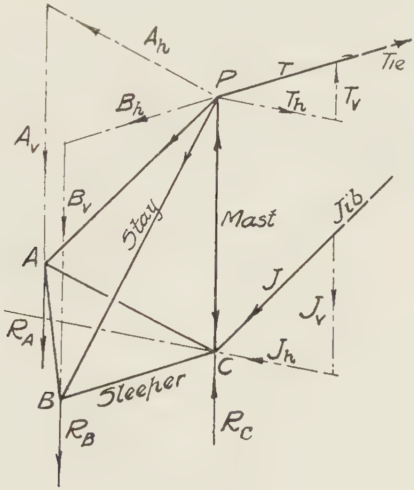


Fig. 93. Position 1.

(*d*) The horizontal component J_h of the force acting in the jib, see Fig. 93, will form, with the forces acting in the two side sleepers, a system of three forces in equilibrium. From this the magnitude of the latter forces may be obtained.

(*e*) The reaction R_C at C will be equal in magnitude to the force acting in the mast together with the vertical component of the force acting in the jib. The total force acting in the mast will equal $A_v + B_v - T_v$. Since the third force is less than the sum of the other two, the resultant force acts downwards; the reaction at C must therefore act upwards.

Position 2. In this position the jib and tie lie in the same vertical plane as the side frame CPB; see Fig. 94.

It should now be clear that, except to ensure stability, the back stay AP and the sleeper AC will not take any part in supporting the load, this being taken entirely by the frame CPB. The readiest means of solution is to draw the true shape of this side frame, the jib and tie, etc., as shown in Fig. 94, and apply graphical methods of solution.

(**Note.** Since flexible stays are not being dealt with it would be unwise to assume that the greatest possible tensile stress in the back stay CP is to be found under the above condition. Other positions on both sides of this one must be tried before a decision can be given. Much will depend upon the relative proportions of the parts of the crane.)

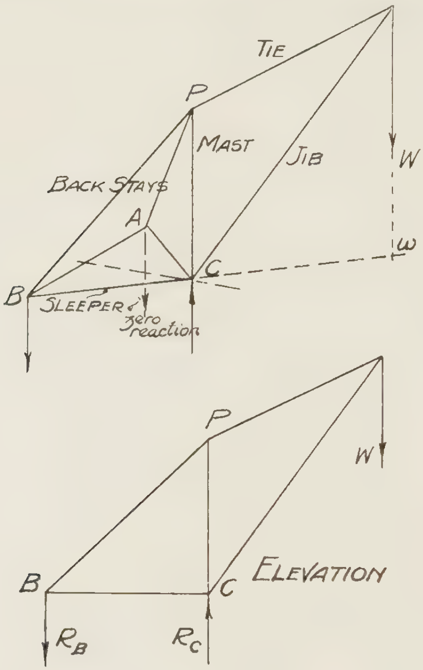


Fig. 94. Position 2.

Position 3 (see Fig. 95). The analysis of this case by geometrical means is interesting but lengthy. The method used for Position 1 should be adopted.

Position 4. As in Position 2 it will be seen that in this case one of the side frames takes no part in supporting the load, the whole of which is taken by the side frame PAC; see Fig. 96. The graphical method suggested for use in dealing with Position 2 may be used here.

It is obvious that the stress in the back stay AP is a compressive stress, which reaches its maximum value in this position.

Only three positions moving in an anti-clockwise direction from Position 1 have been considered. Three other positions in a clockwise direction from Position 1 would be necessary to complete the investigation. Since, however, the crane is symmetrically constructed the values will be the same as those already obtained

but will act in the opposite members to those for which they have already been ascertained.

By plotting the values obtained for the stresses acting in each member for the various positions a series of graphs may be drawn which will indicate the nature and magnitude of the changes taking place. In order to obtain maximum values in the case of some of the members, additional positions may have to be investigated, the same methods being applicable.

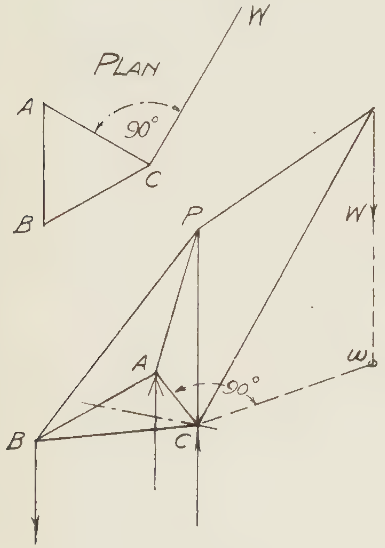


Fig. 95. Position 3.

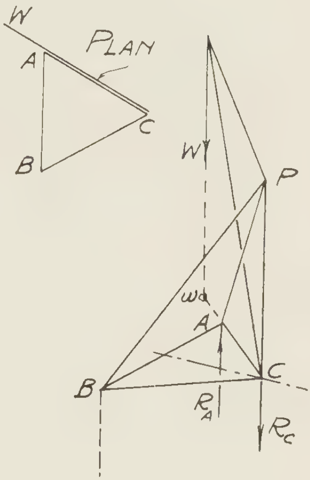


Fig. 96. Position 4.

Derrick Towers

65. The forces acting in derrick towers. By making certain simple assumptions, all of which err on the side of safety, it is possible to use the methods described above so as to obtain an approximate but useful analysis of the forces acting in derrick towers. The same methods can also be applied to other types of large temporary structures used by the builder. It would occupy too much space to deal fully with even one case, but it is hoped that the following outline of the work involved will enable the reader to apply the methods to actual examples. In any case the clearer understanding of the forces acting in these somewhat complicated structures should certainly be of considerable value to those who have to erect and use them.

Construction. A derrick tower or tower gantry consists of three separate towers, masts or legs, corresponding to the three corners A, B and C of the base of a derrick crane; see Fig. 97. The legs are usually constructed of timber framing, being from 5 to 8 ft. square in plan. The leg CE coming under the mast of the crane is known as the "king tower" or "king leg"; since it has to carry the greatest load (see graph, Fig. 66) it is stronger than the other towers. These latter legs are known as the "anchor towers" or "anchor legs", and serve to support the crane at the points A and B and to contain anchor chains, which pass from the sleepers of the crane at A and B to the platforms on which the legs are

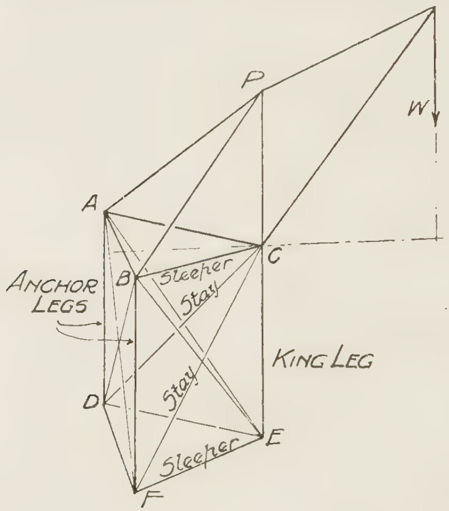


Fig. 97. The derrick tower.

erected. The platforms D and F are suitably loaded (and, if necessary, C also, see Fig. 66) with bricks or other heavy materials so as to be able to resist the overturning forces arising from the effect of the load W and other causes; see para. 48.

The towers may rise to a height equal to two or three times the width of one side. In very tall towers each side frame may be divided by one or more horizontal members into bays. Each of these bays is provided with inclined stays to oppose the twisting and racking forces brought to bear upon the tower. Timber stays are commonly used, but flexible stays are equally suitable.

In order to keep the following analysis on simple lines we will limit our treatment to a case in which no horizontal members are used between the top sleepers and the base of the tower. (In other

cases the stresses in the stays may be found by the methods described in Chap. XI.)

Assumptions. The following assumptions will be made; see Fig. 97:

(i) That the loads are centrally placed on each leg, the legs being looked upon as simple members or "masts", freely jointed at their top and bottom ends.

(ii) That the horizontal members forming the base of the crane are also part of the structure of the tower, all the joints being such as to allow some amount of freedom to the members. (In practice while these joints are certainly not free joints their rigidity is usually of the slightest. In any case our assumption makes for safety.)

(iii) That the stays, of which there are two in each side bay, are only capable of resisting tensile stresses and are only connected to the other members at their top and bottom ends.

Experimental model. It will add considerably to the value of this investigation if each of the statements can be illustrated or checked experimentally as the work proceeds. The model tower described below, which is by no means complicated or difficult to construct, will be found to satisfy all reasonable requirements; see Fig. 98. The framework of the crane surmounting the tower may consist entirely of solid members. It is supported on masts or legs which for simplicity are likewise solid. These are again connected at the bottom by solid sleepers. All the joints should be as simple as possible and such as to allow some measure of freedom of movement.

In plan the tower forms an equilateral triangle, the sides being of equal length.

In each of the side bays there are two stays. These should consist of light chains or of stout cord; in every case there should be some means of adjusting the length (the end may be secured to a hook in the case of the chains, or a small wooden cleat may be used, as in tent stays, in the case of cord). At least one stay in each bay should include a balance in its length so that the force acting in it may be measured. It should be possible to interchange these stays readily so that the stress in any required stay can be measured and that in the other ignored.

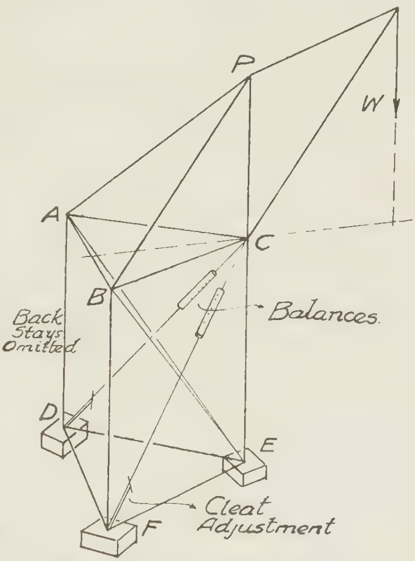


Fig. 98. Experimental derrick tower.

Weighted blocks may be attached at the points D, E and F, and the reactions of the whole structure obtained as described in para. 63.

By adjusting the stays the tower can be brought to a vertical position. In no case should any readings of the balances be taken until this has been done.

66. Outline of the analysis of the forces in the structure of a derrick tower.

I. **System of forces supporting each leg.** Take, say, the king leg *CE* in Fig. 99. Joined to *C* we have two horizontal members, *AC* and *BC*, and two inclined stays, *DC* and *FC*. Whatever forces act in the two top horizontal members they can be replaced by their resultant *R*; see Fig. 99. This force *R*, the leg and the two inclined stays have been indicated in thick lines. It should be obvious from this brief analysis that the mast or leg *CE* is held in position by the action of the horizontal force *R*, the force in the leg, and the forces acting in the two inclined stays. Each leg can therefore be dealt with by the methods outlined in para. 61 provided that sufficient of the unknown forces can be defined.

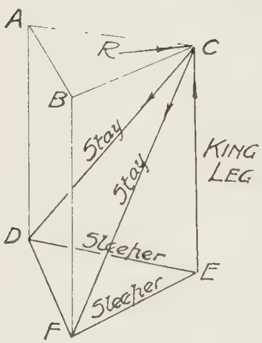


Fig. 99.

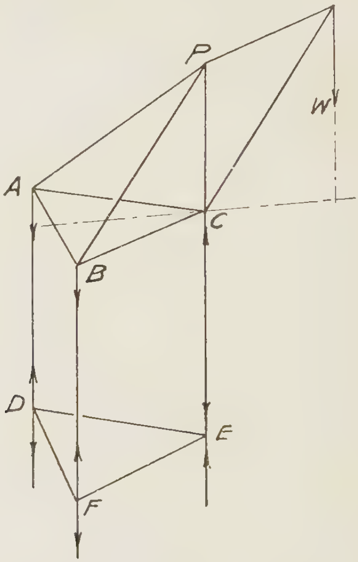


Fig. 100.

II. **Effect of the load *W*.** We have already seen that the reactions of a derrick crane consist entirely of vertical forces (at the points *A*, *B* and *C*), if the only external force applied to the crane is the vertical load *W*. It must therefore follow, if, as we have assumed, each single leg is vertical and the loads transmitted from the crane are applied centrally to these legs, that these vertical forces are transmitted unchanged to the ground; see Fig. 100. If so, then it follows that *the load W does not affect the forces acting in the stays, and the reactions of the entire structure due to the load W consist of vertical forces.* We may test these statements experimentally.

Experiment. *To ascertain the nature of the reactions of a derrick tower and crane when supporting a load W and to find the effect of W on the forces acting in the stays.*

(*a*) The reactions at each of the three points of support should be found experimentally as described in para. 63, when the truth of the statement that the reactions are vertical should be apparent.

(*b*) The fact that the load W does not affect the forces in the stays may be demonstrated as follows: Having erected the tower and crane apply the load W and, by adjusting the stays, bring the tower to the vertical position. To accomplish this, some amount of "initial tension" will be necessary in each of the stays, the magnitude of this force will be the "zero reading" on the balances (about half the stays should include balances).

Now rotate the load W *slowly* and note whether, as the jib takes up the various positions, the readings of the spring balances are altered. *If the legs are all exactly vertical* the results should confirm our first statement.

The purpose of the inclined stays. The question may well be asked, "If the raising, lowering and holding of the load do not affect the forces in the stays, what then is their main purpose?" Obviously they are necessary to maintain the tower in a stable position but, as we shall see, they do in addition render active service when the tower is subjected to what may be called secondary forces, or loads additional to W.

III. **The effect of secondary forces on a derrick tower.** These are forces which, unlike the vertical force W, tend either to push the tower out of the vertical or to twist it about its base and, as will be shown, the stays are the main factors in resisting such forces. The chief secondary forces to be considered are:

(A) **Eccentric loading of the legs.** This produces an effect similar to that arising if the legs are out of the vertical. The effects may be demonstrated experimentally by so adjusting the stays in the model tower that one or more of the legs lean away from the vertical. If the load W be now rotated considerable changes will be noted in each of the stays as the jib takes up various positions. These results, which need not be analysed in detail, emphasise the importance of ensuring that the legs are vertical.

(B) **Effects due to wind pressure.** Any side pressure, such as that due to wind, will tend to push the tower over; its effect will vary and be resisted differently according to the direction in which it acts. Two cases only need be considered here: one in which the wind blows directly upon one face of the tower and the other in which it blows parallel to one face of the tower. In order to simplify our investigation we will make the following assumptions: (i) that there is no shielding of one leg by another, each being equally affected; and (ii) that the pressure due to the wind is uniformly distributed throughout the height of each leg: hence, if P be the total pressure on each leg due to the wind, then we can assume that half this force, or $P/2$, is taken by the joints at the top and bottom ends respectively.

Case I. *Wind blowing at right angles to one face of the tower.* This case is illustrated diagrammatically in Fig. 101 (wind in the reverse direction might also be considered). Those portions of the total wind pressure P which act at the base of the legs may be neglected so far as the structure of the tower is concerned, since they add nothing to the stresses in the members. We are thus left with three forces (each equal to $P/2$) acting at the top of each leg in the direction indicated in Fig. 101. It will be clear from the diagram that the two forces acting at A and B respectively will have two effects: one will be to tend to widen the angle ACB and the other will be to induce a compressive force in the horizontal members AC and CB. The first effect will be resisted by the horizontal member AB, which will be put into tension. The result of the second tendency will be to increase the force acting at the top of the leeward leg CE. Further consideration of the conditions will show that the total equivalent force acting at C will be equal to $3\ (P/2)$; this will be resisted solely by the mast and the two inclined stays attached to the leg at the same point C; see Fig. 101. No other stays will be brought into action.

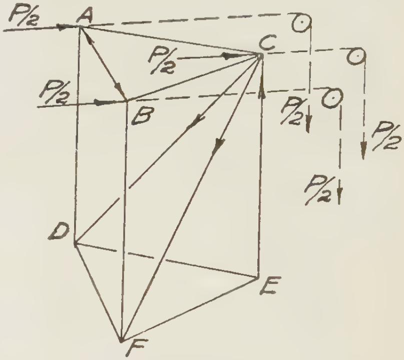


Fig. 101.

The experimental demonstration of the truth of these statements is not difficult. Fig. 101 shows how three parallel forces equal to $P/2$ may be applied at the tops of the three legs in a manner similar to that in which we have taken the wind to be acting. In this way the stresses produced in the stays may be investigated.

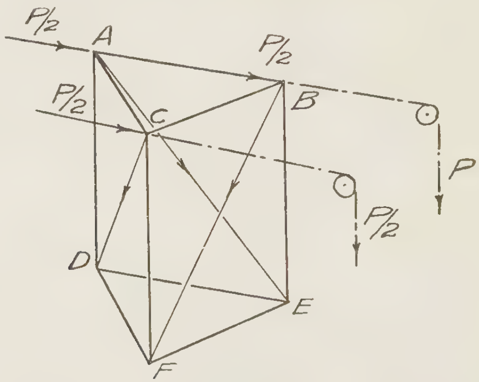


Fig. 102.

Case II. *Wind blowing parallel to one face.* This case is illustrated diagrammatically in Fig. 102, which also indicates how the conditions may be investigated experimentally. The force $P/2$ acting at A is trans-

mitted directly along the horizontal member AB to the point B. The total force P acting at C would be resisted by the stay BD (not shown).

The effect of the force $P/2$ acting at C is not capable of exact analysis by the methods here adopted, but it is clear that its effect is *to twist the top platform of the tower about the base*; this effect will be resisted by all the stays sloping upwards to the right as indicated on Fig. 102. These statements should be checked experimentally.

(C) **Effects due to the swinging of the load.** The third type of secondary effects to be considered are those which arise during the acceleration or retardation of the jib and load as they move from one position to another horizontally—"slewing". The exact value of the forces created in this way are not easily calculated; with careful handling they are not usually considerable in amount. The effects are similar to those mentioned in the last case, viz. to twist the top platform about the lower one, see Fig. 103; this will be resisted by the three stays inclined upwards in the direction in which the force tends to twist the tower. The experimental demonstration of this case may be carried out as suggested in Fig. 103.

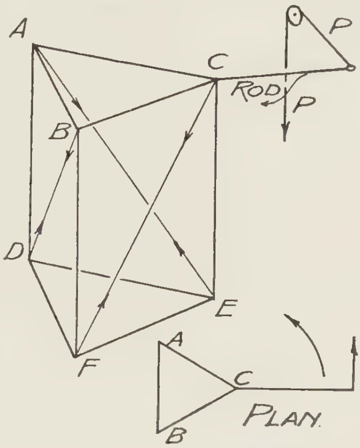


Fig. 103.

IV. **The overturning effects of the wind.** The effect of a side pressure, such as that due to wind, on a derrick tower as a whole will be to produce an overturning moment. This will be resisted by the stability moment due to the weight of the tower; see para. 48. Obviously the effect will be to induce certain reactions at the base of the tower. The worst case usually arises when the wind is acting in the direction shown in Fig. 104, since, when the crane is not in use, the weight acting downwards in the king leg is not usually considerable and, as will be seen from the sketch, this is the weight upon which the stability of the tower will mainly depend. The problem should be investigated for all towers, but particularly for those which are of exceptional height or which have to be erected in exposed positions. As the following example will show, a solution can be obtained which is sufficiently accurate for most practical purposes, by the application of the analysis outlined above.

Example. *The derrick tower shown in Fig.* 104 *is* 60 *ft. high, the sides being each* 30 *ft. wide. It is calculated that, using a pressure of* 30 *lbs. per sq. ft. on the exposed surface, the total pressure* (P) *exerted on a single leg is approximately* 2500 *lbs. If this pressure may be distributed over the whole structure in the manner indicated in Fig.* 104, *find* (*a*) *the stress in the inclined stays* AF *and* BF, *and* (*b*) *the reactions induced at points* D, E *and* F. *Check the latter results by equating the overturning and stability moments.*

The force of $P/2$ lbs. acting at C will be transmitted along the members CA and CB to points A and B, where it may be resolved into its rectangular components along the member AB (with which we are not

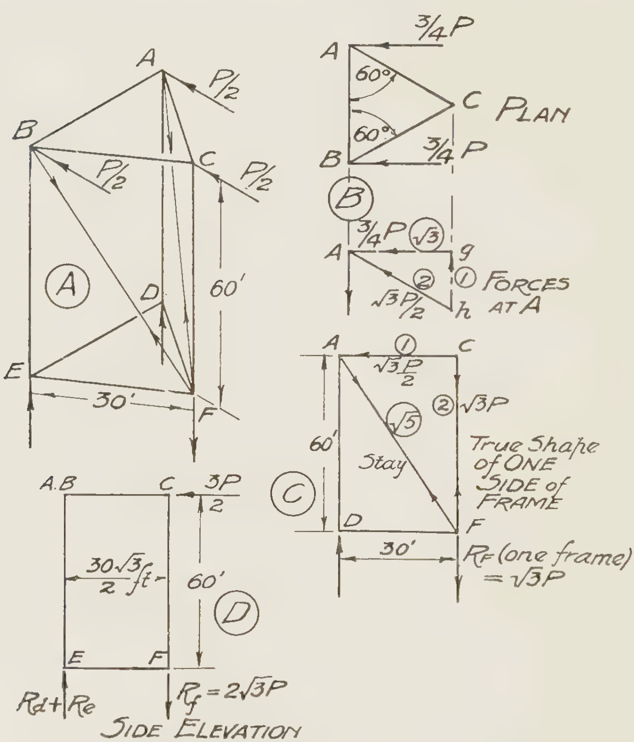


Fig. 104. Stability of a derrick tower subjected to wind pressure.

further concerned), and along the lines in which the other two forces $P/2$ are already acting. It is easy to show that we thus arrive at the condition indicated in the plan; Fig. 104 (B). The two forces $3P/4$ acting at A and B may likewise be resolved into their components along AB, AC and BC. See small force triangle Agh, in which the ratios of the sides have been indicated by figures in small circles, the figure being half an

equilateral triangle. From this we have that the force acting at A in line AC is given by Ah in this triangle and equals

$$\left(\frac{3P}{4} \times \frac{2}{\sqrt{3}}\right) \text{ or } \frac{\sqrt{3}P}{2}.$$

In Fig. 104 (C) the shape of one of the side panels has been set out. Since the sides of the triangle ACF are parallel to the forces which we are to consider we may use it as a force triangle. Thus the horizontal force acting in AC of $\frac{\sqrt{3}P}{2}$ is the horizontal component of the force in the stay. Using the small figures in circles (which give the ratios of the sides), we have

$$\text{force in stay } AF \text{ (= force in stay } BF) = \frac{\sqrt{3}P}{2} \times \sqrt{5}$$
$$= 4840 \text{ lbs.}$$

Again, using the same force triangle, the vertical component of the force in the stay is, from the proportions of the force triangle, twice the magnitude of its horizontal component, so that

$$\text{the upward acting force at } F = 2\,\frac{\sqrt{3}P}{2} = \sqrt{3}P.$$

But this is due to one stay only, so that with both stays equally stressed the total upward force at $F = 2\sqrt{3}P = 8660$ lbs.

To check by overturning and stability moments; see Fig. 104 (D).

The overturning moment is evidently equal to $3P/2 \times 60$.

Since the perpendicular distance between the towers in side elevation may be shown to be $(30 \times \sqrt{3}/2)$, therefore the stability moment, taking moments about E, is (reaction at F) $\times \frac{30\sqrt{3}}{2}$.

Equating the two moments we have

$$R_f \times \frac{30\sqrt{3}}{2} = \frac{3P \times 60}{2},$$

from which we have

$$\text{reaction at } F = 2\sqrt{3}P = 8660 \text{ lbs., as before.}$$

The reactions at D and E being equal, the magnitude in each case will be half this or 4330 lbs.

When it is noted that *the force at* F *must act as a holding-down load for stability,* it will be seen how important it is to check the total weight on the king leg to see whether any additional weight must be added.

Problems VII

1. If in Fig. 83 the force in the stay BP is 10 lbs. and the angle θ is 30°, the lines of action of the stays being equally inclined in plan; find (a) the forces acting in A and C, and (b) the force in the mast.

2. Find the force acting in the mast in Fig. 84, if all the stays are equally inclined at 60° to the horizontal and the tension in each of them is 10 lbs.

3. Find the forces acting in the horizontal stays at PA and PC in Fig. 87, if the angle BPW is 30° and the pull W is 1000 lbs.

4. Find the forces acting in the horizontal stays PD and PA in Fig. 88, if the angle CPW is 30° and the pull W is 1000 lbs.

5. If in problem 4 the lines PD and PA represented the plan of stays each inclined at 45° to the horizontal, find the forces acting in the stays for the same horizontal pull of 1000 lbs.

6. In the mast derrick with four stays symmetrically arranged, shown in Fig. A, if a load of 5 tons is swung at a radius of 30 ft., find, for the given dimensions, (a) the vertical uplift at A, (b) the vertical

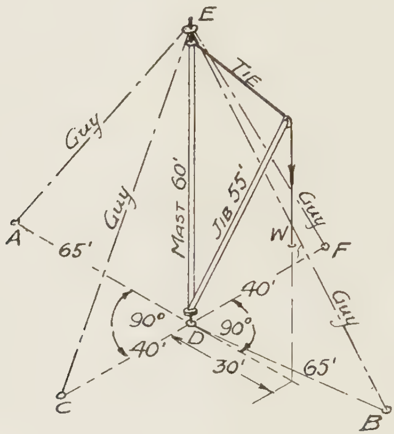


Fig. A. Stayed-mast derrick crane.

and horizontal reactions at D, (c) the force acting in the stay AE, and (d) the force acting in the mast. The stay AE may be assumed to lie in this case in the plane containing the mast and the jib of the crane on the opposite side, the stay AE supplying the whole inclined force necessary for stability.

7. Using the dimensions given in the Example in para. 66 (IV), and P equal to 2500 lbs., find the stress in the stays sloping down from C, and the three reactions at D, E and F when the wind pressure acts as shown in Fig. 101.

8. If in the crane illustrated in Fig. A, the plane of the jib bisects the angle made by the two stays CE and BE, find for the same load the stresses acting in the two opposite stays AE and EF. The stays are symmetrically arranged on plan, but the points C and F are only 40 ft. from the base of the mast.

SECTION II

LOADED BEAMS AND THE THEORY OF ELASTIC BENDING

CHAPTER VIII

THE EQUILIBRIUM OF FORCES IN A LOADED BEAM

67. Introduction to Section II. The beam has always been an important and essential element in the structures erected by man. In its earliest form it would consist of a log of wood or a rough slab of stone and be used to span an opening for the purposes of shelter, transport or defence. With time its use became better understood and its form refined. Thus it was that in all the great buildings of classical times the beam was found in the highly decorative forms which have come down to us in the architectural styles of those periods.

Experience soon made it evident that there were very distinct limitations to the size of the opening which could be spanned by the beam, especially when it had to be made of a material such as stone, which is relatively weak in resisting tensile stress. With the invention and development of the arch therefore, the beam came to play a less prominent part in all permanent structures until recent times.

The great and comparatively recent developments in the use of iron, steel and reinforced concrete have, however, once more restored the beam to a position of first importance in all modern fabricated structures. Thus it is incumbent upon all who occupy responsible positions in connection with the design and erection of buildings and who have to cope with the practical and technical problems of modern building, that they should possess some knowledge of the actions which take place in the material of a loaded beam and of the factors which control its design.

The theory of elastic bending has grown out of the labours of many physicists, mathematicians and other investigators since the time of Galileo in the sixteenth and seventeenth centuries. Yet in spite of the relative complexity of the facts with which this theory deals, and because of its development, it is possible to include in this volume a fairly simple explanation of the actions taking place within a loaded beam. The greater part

of this section will be devoted to this explanation, together with some account of the more important practical applications to be found in building work.

68. The Beam. The beam is essentially a structure which spans one or more openings and which carries, in addition to its own weight, a load or series of loads applied transversely to its length. With the accumulation of experience and the growth of knowledge the construction of the beam has been elaborated so that there are now quite a variety of types; these may be briefly enumerated as follows:

Types of Beams. The term "beam" may be applied to a single horizontal member supported at each end, see Fig. 105; it may likewise be applied to an inclined member, see Fig. 106; such beams may have other sections than a simple rectangular one, see Fig. 107; a beam may

Fig. 105.

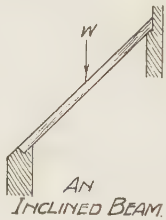


Fig. 106.

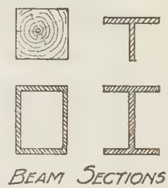


Fig. 107.

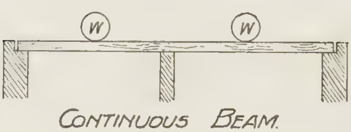


Fig. 108.

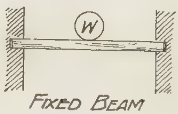


Fig. 109.

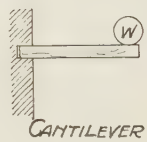


Fig. 110.

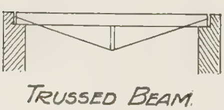


Fig. 111.

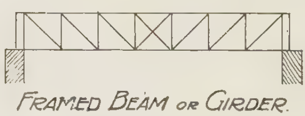


Fig. 112

be continuous over several openings, when it is known as a **Continuous Beam**, see Fig. 108; a simple beam may be fixed at each support, when it becomes a **Fixed Beam**, see Fig. 109; if on the other hand it is only fixed at one end and unsupported at the other it is known as a **Cantilever**, see Fig. 110; a simple beam may be strengthened by trussing, when it is called a **Trussed Beam**, see Fig. 111; while large and important beams may consist of a framework, when they are known as **Framed Beams** or **Girders**, see Fig. 112 (in this sense any framework spanning an opening, such as a roof truss, is a framed beam).

69. General effects of a load on a beam. It is convenient to deal with the problem of the loaded beam in two stages. Proceeding along lines which should now be quite familiar to the reader we will, in the first stage, endeavour to ascertain the relations which exist between the external forces applied to the beam and the resulting internal forces, which must be called into action in the material of the beam to produce equilibrium.

In the second stage—having briefly investigated the property known as **elasticity**, which is possessed in varying measure by all structural materials—we shall endeavour to ascertain what are the effects produced in the material of the beam when it is bent or strained, and in what manner these effects are related to the internal forces which, in the first stage of our enquiry, we shall have shown to be necessary.

If, however, we are to derive the fullest possible benefit from this order of procedure it is necessary that we should first obtain some general conceptions of the manner in which a beam is strained when it is loaded. The following simple experiments will illustrate these effects in sufficient detail for our present purpose.

I. External effects—Bending. If a load is placed upon a beam it will bend. This bending may be so slight as to be imperceptible to the unaided eye. The form which the beam assumes is, however, quite definite and—assuming that the beam is of uniform section throughout—depends solely upon the manner in which the beam is loaded and supported. Some of the forms which loaded beams assume are shown to an exaggerated scale in Figs. 113 to 116. They may be illustrated experimentally by using a light rod or bar of wood or metal. The following cases at least should be observed:

Experiment. (*a*) **Simple or freely supported beam.** Support the beam on two benches and load as shown in Fig. 113. Note that the beam bends to a single curve.

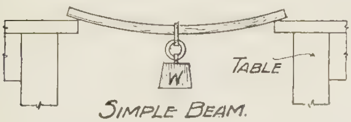


Fig. 113.

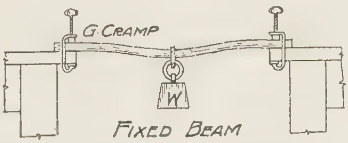


Fig. 114.

(*b*) **Fixed beam.** In this case the two ends of the beam are clamped as shown in Fig. 114. Note that, while the beam is horizontal at each end, there is a reversal of the curvature between the weight and each support.

(*c*) **Cantilever.** Only one end of the beam is fixed while the other is free and supports the weight; see Fig. 115. Note that the cantilever is horizontal at the fixed end and bends to a single curve for the remainder of its length.

(*d*) **Continuous beam.** Arrange the beam—which should be fairly long in this case—as in case (*a*) but add a support at the centre; see Fig. 116.

Two equal loads should be applied. Note that the beam is horizontal over the centre support. At the free ends the shape is similar to that taken up by the simple beam.

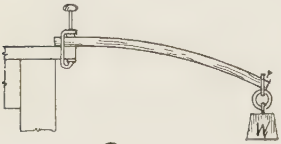


Fig. 115.

Fig. 116.

II. Internal effects – Tension and compression. Take a short deep beam of rubber, see Fig. 167, and mark on one side of it a series of equally spaced vertical lines. Now, by applying pressure or by adding a weight, depress the centre of the beam when it is supported at each end, see Fig. 167, and note that the space between each of the vertical lines, and especially near the centre, is reduced at the top and increased at the bottom. From this it should be obvious that the upper fibres of a bent beam are compressed (compression) while the lower fibres are stretched (tension).

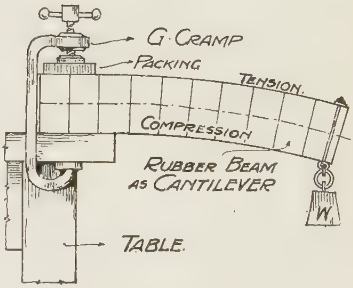


Fig. 117.

It may be noted here that, in a cantilever, the effect of the load is to stretch the upper fibres and to compress the lower ones, see Fig. 117 (which also shows how the same rubber beam may be used experimentally as a cantilever).

Note. The same effects as those described above may be demonstrated by noting what happens to sawcuts made in the upper and lower edges of a wooden beam.

III. Internal effects – Shear. Another and less obvious effect arising out of the loading of a beam is the tendency for one portion to be pushed past or "sheared off" the adjoining portion of the beam; thus in the cantilever shown in Fig. 118 at any section CD the load W, acting vertically, tends to shear the outer portion from the inner along the plane of this section; it should be clear that some force or forces must be called into action by the straining of the beam to resist this tendency.

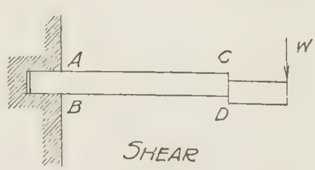


Fig. 118.

70. The equilibrium of a portion of a loaded cantilever. It is convenient to deal first with the case of a loaded cantilever, since it gives in the simplest form all the factors present in a bent beam.

In the cantilever ab, shown in Fig. 119, which has a span of L units, depth y units and carries a load W at the end, let us imagine that the portion $abcd$ is separated at the vertical plane ad

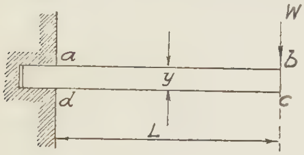


Fig. 119.

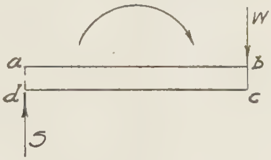


Fig. 120.

from the fixed portion of the cantilever on the left. We may thus consider what forces must act at this section to maintain the equilibrium of this portion $abcd$ when acted upon by the load W; see Fig. 120.

Ignoring the weight of $abcd$ the only external force acting upon it is the force W. Let us therefore assume that a vertical force S, acting at the section ad, is supplied by the material of the beam. *For equilibrium of the vertical forces* this must act upwards and be equal to W. This force S evidently resists the shearing action of the load to which we have already referred. There are thus two vertical but opposite forces acting on the portion $abcd$ at a distance equal to L. Such a pair of parallel forces is known as a **Couple** (the general characteristics of couples are discussed in para. 72). The effect of these forces, if unresisted in other ways, would be to rotate $abcd$ in a *clockwise* direction as indicated in Fig. 120. Clearly then, to produce equilibrium, this tendency must be resisted. Let us therefore consider further the equilibrium of $abcd$.

Taking moments about any convenient point in ad, say d, see Fig. 121, we find that the moment of the force S about this point is zero, since the line of action of S passes through d. The only other force on the beam is W and its moment about d is equal to $W \times L$, acting in a clockwise direction. To balance this moment assume that there is a horizontal force T acting at a and having an anti-clockwise moment of $T \times y$ about d. Then for equilibrium about d, we must have

$$T \times y = W \times L,$$

whence
$$T = \frac{W \times L}{y}. \qquad \text{......(a)}$$

If we consider next the equilibrium of *abcd* about some other convenient point in *ad*, say *a*, see Fig. 122, then both S and T will have zero moment about this point and we have as before, the

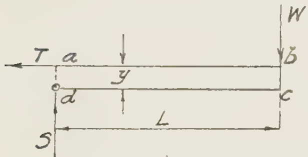


Fig. 121.

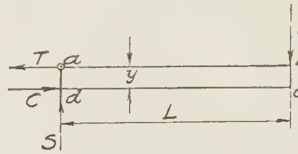


Fig. 122.

single force W, which has a clockwise moment about the point a. To balance this moment assume that there is a horizontal force C acting at d, with an anti-clockwise moment about a. Then for equilibrium about a we must have

$$C \times y = W \times L,$$

whence $$C = \frac{W \times L}{y}. \qquad \text{......(b)}$$

Comparing (a) and (b) we see that since

$$T = \frac{W \times L}{y} = C,$$

$$\therefore \quad T = C.$$

These two forces T and C clearly form a second "couple", which tends to turn *abcd* in an *anti-clockwise* direction.

If we can show that *abcd* is in equilibrium under the action of the four forces which we have now defined, then the couple formed by the two forces T and C must be sufficient to balance the couple formed by the two forces W and S.

Since the vertical forces W and S are equal in magnitude and opposite in sense it follows that the sum of the vertical forces is zero or $\Sigma V = 0$.

Again, since the horizontal forces T and C are equal in magnitude and opposite in sense, it follows that the sum of the horizontal forces is zero or $\Sigma H = 0$.

Finally, taking moments about any point, which for convenience can be d (or c), we have already shown that

$$W \times L = T \times y,$$

hence $$W \times L - T \times y = 0.$$

In other words, the sum of the moments of these forces about any point in the plane containing them is zero or $\Sigma M = 0$.

We thus find that all the general conditions for the equilibrium of a number of coplanar forces have been satisfied, from which it follows (a) that the two couples balance each other, and (b) that

the portion *abcd* of the cantilever is in equilibrium under the action of these four forces.

71. Experiment. Before proceeding further it will be helpful if we can carry out an experiment to check the conclusions at which we have

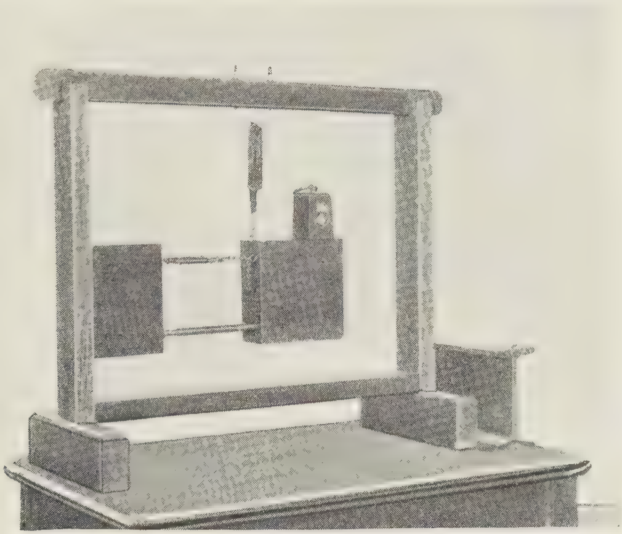


Fig. 123. Experiment on a cantilever.

just arrived. The apparatus which may be used is illustrated in Fig. 123, and shown diagrammatically in Fig. 124.

The block *abcd* represents that portion of the cantilever which has been separated from the remainder of the cantilever at the section *ad*. In the apparatus this block is supported vertically at the point *a* by a spring balance *S*; this balance can be raised or lowered or moved horizontally along the support *GH*, which is slotted for this purpose, the balance measuring the vertical force *S* acting at *a*.

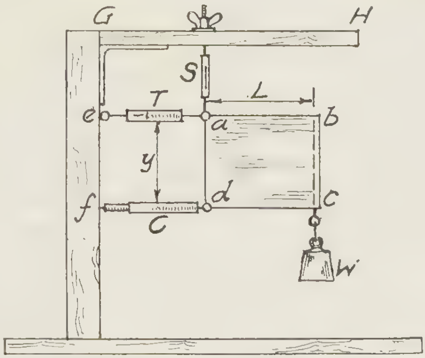


Fig. 124. Equilibrium of a portion of a cantilever

The points *a* and *e* are connected by a pair of balances lying side by side; the distance between these two points can be adjusted by means of an adjusting screw or a short piece of chain. These balances measure the horizontal force acting at *a*.

The points *d* and *f* are connected by a compression balance so that the force acting at *d* can likewise be measured.

As the apparatus has weight it is necessary to allow for this; the procedure is as follows:

(*a*) By altering the distance *ae* and the height of *a*, adjust the block until the points *a* and *b* lie on a horizontal line passing through *e*. Read off the balances S, T and C; record these readings as the "zero" readings.

(*b*) Attach the load W at the point *c*, or place at some convenient position on the top of *abcd*, and once more adjust *a* and *b* to lie on a horizontal line through *e*. The *increase* in the readings of the three balances will give the magnitude of the forces called into play by the load W at the point *c*.

If the experiment has been carefully carried out, it should be found that $S = W$, $T = C$ and, if the distances L and y are measured, it should also be found that $W \times L = T$ (or C) $\times y$.

The experiment will therefore have demonstrated the fact that the body *abcd* is in a condition of equilibrium under the action of two vertical forces, W and S—which together form a couple with a clockwise moment—and two horizontal forces, T and C—which together form a couple with an anti-clockwise moment—and that these two couples balance.

72. Couples. It is convenient at this stage to consider more fully the chief characteristics of Couples.

(*a*) *A couple consists of two equal but opposite forces acting along parallel lines*; see Fig. 125 (A).

(*b*) When applied to a rigid body a couple tends to make that body *rotate* in a clockwise or anti-clockwise direction.

(*c*) *The magnitude of a couple is given by the product of the magnitude of one of the forces into the amount of its perpendicular distance from the line of action of the other force.* Thus the magnitude of the couple shown in Fig. 125 (A) is given by $F \times d$. The distance d is known as the "arm" of the couple.

(*d*) *The effect of a couple is unaltered by any change in the position of the couple, provided that it remains in the same plane.*

To illustrate this fact consider the application of the couple Fx in the position (1) shown in Fig. 125 (B); if moments be taken about any point Z, at a distance r from the nearer force, then the magnitude of the resultant moment of the two forces about this point is given by

$$F\,(r + x) - Fr,$$

or $Fr + Fx - Fr$ which equals Fx (the magnitude of the couple as first defined).

Similarly, if the same couple be applied in a new position, say (2) in Fig. 125 (B), we have the resultant moment given by

$$F\,(s + x) - Fs,$$

which reduces to Fx as before, and similarly for any other position.

(*e*) *A body will be in equilibrium if acted upon by two couples of equal magnitude but opposite effect.* This can always be shown to be true by taking moments about any point in the plane containing the two couples.

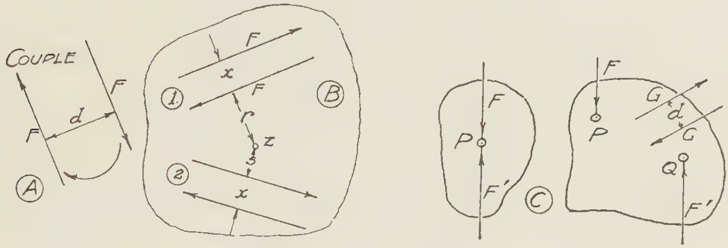


Fig. 125. Couples.

(*f*) Let a force F act at the point P in a body as shown in Fig. 125 (C); then it is possible to produce equilibrium by the application at P of a second force F', of equal magnitude but opposite sense as shown. Suppose that, for some reason, it is not possible to apply the second force F' at P, but that it must be applied at some other point, say Q. Then the two parallel forces (F and F') will constitute a couple and may be balanced by a second couple, say Gd, of equal magnitude but of opposite effect to the first couple. We thus see that *a single force may be balanced by a couple plus a single force.*

Experiment. The truth of the above statements may be demonstrated experimentally in the following manner. A rod, which should be fairly long and heavy, is suspended from two spring balances S and T as shown in Fig. 126 (A). When the rod is horizontal the pulls are noted and recorded as "zero readings".

(1) A couple is now applied to the rod by adding a downward acting force P at B and an equal upward acting force at A. Obviously when the rod is again made horizontal, the reading in each balance will have been increased by an amount equal to the magnitude of P. The magnitude of the applied couple will be given by $P \times AB$.

Now remove the two forces from A and B. Apply the upward acting force at some other point, say C; see Fig. 126 (B). Suspend the downward acting force from the rod and move it about until, if possible, the balances S and T indicate equal increases of the same amounts as before. Measure the distance x. The distance x should be found to be equal to the distance AB.

(2) The second part of the above experiment should now be repeated, using two equal weights of greater or less magnitude than P, when it should be found that, providing the applied couple has a magnitude equal to ($P \times AB$), the balances S and T will register the same increases as before.

(3) In either (1) or (2) we can show that the rod, when at rest, is in equilibrium under the action of two equal but opposite couples.

(4) Finally, the attempt should be made to produce equal increases in the two balances by applying a single force at any point along the rod. This will be found to be impossible, thus demonstrating that a couple cannot be replaced by a single force.

Other experiments on couples may be carried out in the manner suggested in Fig. 127, using the apparatus shown in Fig. 124.

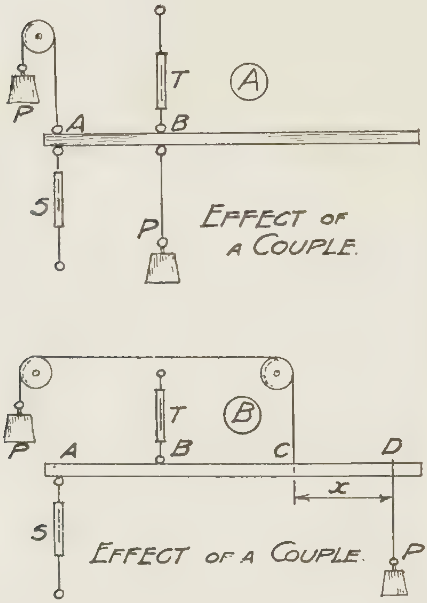


Fig. 126.

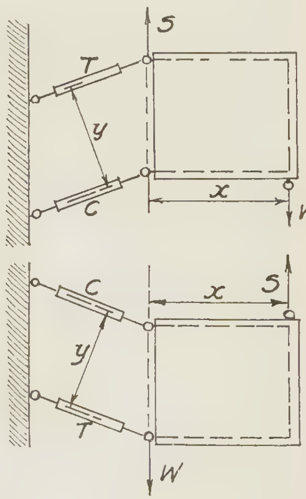


Fig. 127.

Experiments on Couples.

73. The balance of forces in a loaded beam. Considering once more the forces acting on a portion of a loaded cantilever, see Fig. 128, it will be seen that, while W represents the only external force acting to the right of the section ad, the forces acting within the dotted circle, which, as already found, are capable of producing equilibrium, must clearly correspond to the internal forces which are supplied by the strained material of the beam. It should be noted that in this case these internal forces are equivalent to a single force and a couple. This is always so, except in certain special cases when either the force or the couple is reduced to zero value; see para. 88.

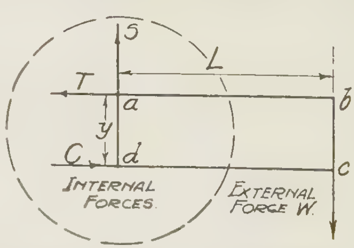


Fig. 128. External and internal forces in a cantilever.

From what we already know concerning couples, it is evident that the exact position or manner of application of the internal couple is, for our present purpose, immaterial, provided that its magnitude remains unaltered. We may therefore proceed at once to summarise our results, knowing that the truth of them is unaffected by our present lack of knowledge of the exact distribution of the strains in the material of the beam, and which are responsible for the production of the forces within the beam.

If W be the resultant of all the forces to the right of the section ad, then $W \times L$ is the external moment applied to the beam; this moment will now be referred to as the **External Bending Moment** (B), since, as we shall see later, its effect is to *bend* the beam.

Now for equilibrium the magnitude, or moment, of the internal couple, which we will now call the **Resistance Moment** (R), must be equal in magnitude to the bending moment, or

The external bending moment = resistance moment, ...(i)

from which it follows that $W \times L = T \times y = C \times y$.

Again, as we have seen, $T = C$, but evidently

$T =$ total tension in the material of the beam at the section ad, while $C =$ total compression at the same section;

hence **Total tension = total compression.** ......(ii)

Finally, as we have seen, $S = W$, but S is the total internal shearing stress at section ad, while the total external shearing force at the same section is equal to W, then

Total internal shearing stress = total external shearing force. ...(iii)

The full significance of these statements may not be apparent to the reader until he has learnt more about the conditions existing in the materials of a loaded beam. The above summary will, however, be found helpful at this stage in assisting the reader to grasp the relations which must exist between the external and internal forces producing equilibrium in a loaded beam.

74. Definitions of Shearing Force and Bending Moment. To ascertain the magnitude of the external forces tending to produce shear and bending at any section, it is immaterial whether the forces to the right or to the left of that section are considered, since, from the necessary balance of the external forces about that section, we find that the same *numerical* result is obtained in each case. Thus, if the forces to the right of a section result in a downward acting force of magnitude W, there must, for equilibrium, be an equal and opposite force of W acting upwards to the left of the section. Similarly, if forces to the right of the section result

in a clockwise bending moment of magnitude B, there must, for equilibrium, be an anti-clockwise moment of magnitude B produced by the forces on the other side of the section. These considerations enable us to define shear and bending moment; they also indicate the need for some convention as to sign, since it is frequently necessary to obtain the "algebraic sum" of a number of forces or of moments.

External Shearing Force. The total external shearing force at any section of a beam is given by the algebraic sum of all the vertical forces acting either to the right, or to the left, of that section.

Sign. External shearing forces will be considered to be positive if they act downwards (or clockwise) to the right of the section. ......(iv)

Bending Moment. The total external bending moment acting at any section of beam is given by the algebraic sum of the moments of all the forces acting to the right, or to the left, of that section.

Sign. The bending moment will be considered to be positive if it acts in an anti-clockwise direction to the right of the section. ......(v)

A convenient way to remember the above conventions is to consider the case of a simple beam supported at both ends and loaded at the centre; see Fig. 129. The beam bends to a concave curve on the upper side and the bending moment is *positive* throughout the length of the beam. That the bending moment is positive is readily seen by considering any two sections c and d, at each of which the bending moment (B) is seen to be anti-clockwise.

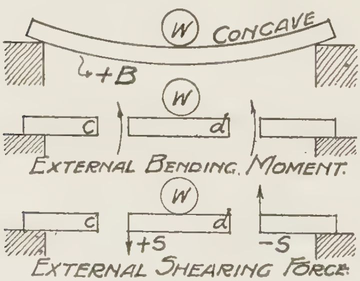


Fig. 129.

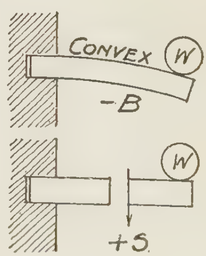


Fig. 130.

In this case, if the shear at the same two sections is considered, it will be seen that the shear is positive to the left of the load W and negative to the right; see Fig. 129.

In the case of the cantilever, see Fig. 130, the shear is positive and the bending moment negative throughout the length. In this case the beam bends to a convex curve on the upper side.

Units. Shear Force. According to the units of weight used, then shear force will be measured in lbs., tons, kilogrammes, etc.

Bending Moment. According to the units of weight and length used, then bending moment will be measured in lb. ins., ton ins., ton ft., kilogramme metres, etc.

Problems VIII.

1. If y in Fig. 124 measures 6 ins., L is 9 ins. and W is 10 lbs., calculate the magnitude of S, T and C. What is the magnitude of B, the bending moment at the section ad?

2. What is the magnitude of the couple applied to the rod in Fig. 126 (A) if P is 8 lbs. and AB measures 6 ins.? If two weights of 12 lbs. each are applied at C and D, as shown in Fig. 126 (B), what will be the distance CD?

3. If, in Fig. 128, W is 1000 lbs. and L measures 5 ft., what is the magnitude of the single force, and the magnitude and direction of the couple, which are applied at the section ad to produce equilibrium?

4. If, in Fig. 124, L measures 12 ins. and four weights of 4 lbs. each are placed at distances of 3, 6, 9 and 12 ins. respectively from a; (i) What will be the sign and magnitude of the shearing force (S) at ad, and (ii) What will be the sign and magnitude of the bending moment (B) at the same section?

5. In Problem 4, what single weight would produce the same shear force (S) and the same bending moment (B) at the section ad, and where would it have to be placed?

CHAPTER IX

SHEARING FORCES AND BENDING MOMENTS IN BEAMS

75. Diagrams of Shearing Force and Bending Moment. In studying the loading of beams it is necessary to consider how the shearing force and the bending moment vary from point to point along the beam. These variations can be conveniently shown by means of diagrams or graphs, drawn with a base corresponding to the length of the beam and having vertical ordinates which, to a suitable scale, give the magnitude of shearing force or of the bending moment at every point along the beam. For all the simpler cases of loading and fixing, the regular forms which these diagrams take can be memorised readily; to complete the diagrams it is then only necessary to ascertain the magnitudes of the principal ordinates. These magnitudes can be obtained either by mathematical or by graphical methods. In some of the more complex cases a combination of the two methods may be necessary.

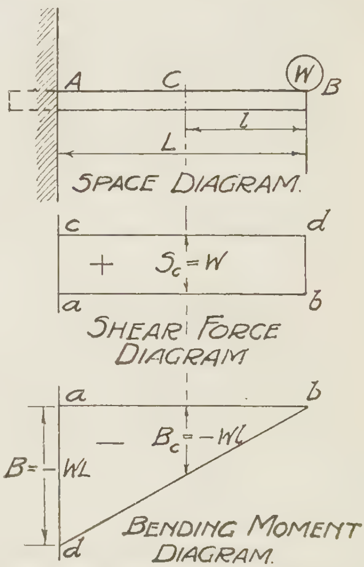


Fig. 131. Cantilever with single load.

Note. In each of the following examples the load may be measured in lbs., tons, or some other convenient unit of weight, while all the distances may be measured in inches, feet, or some other convenient unit of length. For the sake of simplicity we shall for the present work all examples in lbs. and ins., hence moments will be measured in units of lb. ins.

76. Cantilever with a single load at the end.

Shear Force Diagram. See Fig. 131. At any section C the external vertical force, tending to shear the outer portion of the cantilever from the inner, will be *positive* and equal to the magnitude of W; or

$$S_c \text{ (the shear force at } C) = W.$$

From this expression, we see that the magnitude of the shear force will be the same at every point along the cantilever; the shear

force diagram will therefore be a rectangle, having a base *ab* equal in length to the length of the cantilever (drawn to the linear scale), and of such a height *ac* that, to a suitable force scale, the ordinates represent the value $S = W$.

Bending Moment Diagram. See Fig. 131. At any section C, at a distance l from B, the magnitude of the bending moment will be $W \times l$, acting clockwise, or

$$B_c \text{ (the bending moment at } C) = -(W \times l).$$

At B the value of the bending moment will obviously be zero. At point A it will reach a maximum and be equal to $W \times L$. If a number of intermediate points are taken and the values of the bending moments plotted to a suitable scale, the ends of the ordinates will be found to lie on a line drawn through b and d, where ad represents, to a suitable scale, the value of the bending moment at point A. The diagram of bending moment for this case is thus a triangle, a fact which could have been deduced mathematically from the form of the expression, which indicates that the magnitude of the bending moment increases directly with the distance measured from the outer end B.

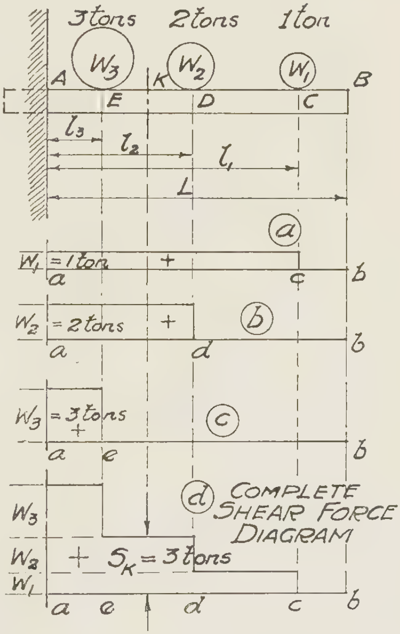


Fig. 132. Cantilever with several single loads—Shear force.

77. Cantilever with several single loads—Addition of diagrams.

Let the loads and points of application be as shown in Fig. 132.

Shear Force Diagram. It will be convenient to consider, in the first place, the effect produced by each single load, as shown by the three diagrams (a), (b) and (c). Between the points C and B it should be obvious that there is no shear force, since there is no load beyond the point C, and similarly for the length to the right of each load, for which separate diagrams are drawn.

The addition of these diagrams to form the single diagram (d)

ought now to be clear. The way in which the value of the shear force increases at each point where an additional load occurs, should be noted. The total value of the shear force at any point K will be given by the corresponding ordinate drawn on the complete shear force diagram, as shown in Fig. 132 (d).

Bending Moment Diagram. We may now proceed in the same way with the construction of the bending moment diagram. The separate diagrams for each load are shown in Fig. 133 (a). When these are added they give the complete bending moment diagram outlined in Fig. 133 (b). The magnitude of the bending moment at any point K is shown by the corresponding ordinate drawn on the complete diagram and measured to the selected bending

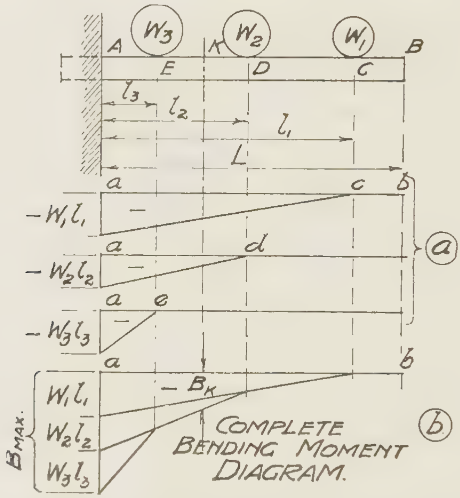


Fig. 133. Cantilever with several single loads—Bending moment.

moment scale. It may be noted here that there are no sudden steps in the bending moment diagram, but that each additional load has the effect of *increasing the inclination of the outline* of the diagram as we approach the point of support.

78. Scales. In setting out these diagrams it will be necessary to select three convenient scales: the **Linear Scale**, the **Force or Shear Scale** and the **Bending Moment Scale.** After these have been selected they should be clearly noted on the drawing sheet in close proximity to the diagrams. The Linear Scale will usually correspond to the linear scale to which the beam has been drawn. The Force Scale will be chosen to represent so many tons or lbs. to the inch, while the Bending Moment Scale will represent so many

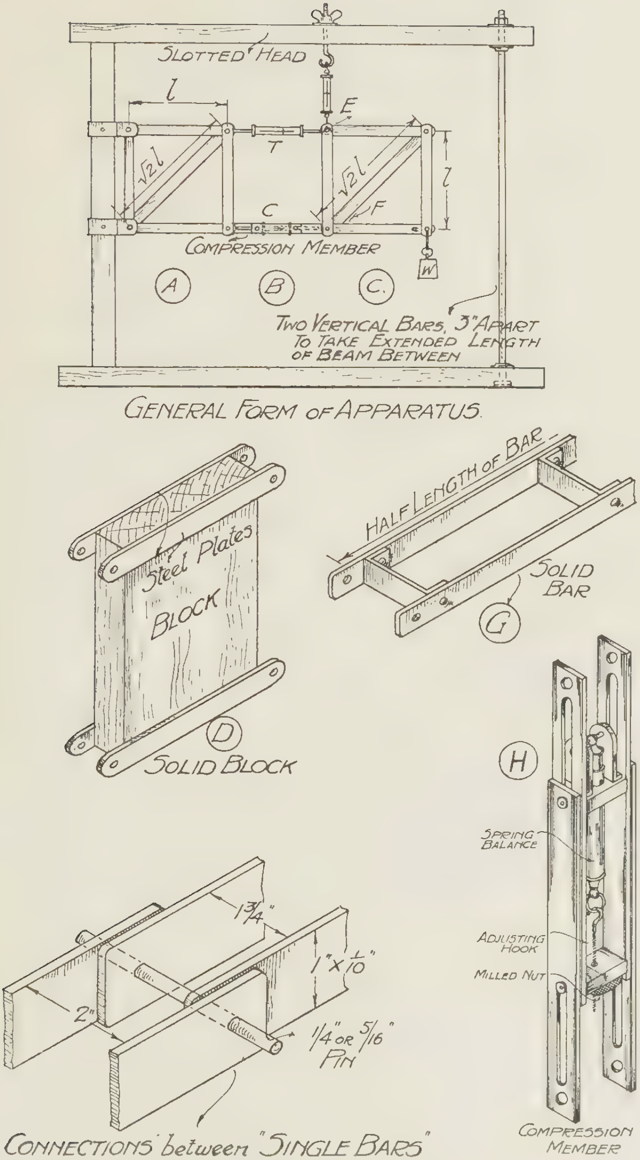


Fig. 134. Experimental cantilever.

"ton ft", "ton ins." or "lb. ins." to the inch according as tons or lbs. and feet or inches have been selected for the force and linear scales respectively. As indicated in the diagrams already drawn, positive values of shear and bending moment will be set up above the base line and negative values below.

79. Experiment. *To find the forces acting at various sections in a cantilever*; see Fig. 135.

To carry out this and other experiments on beams, a piece of apparatus designed by the authors may be used; see Figs. 134 and 134 A. This enables the forces to be found acting over several portions or "bays" of a cantilever, and may be utilised later to find the forces acting in the members of framed girders; see also Fig. 157 A.

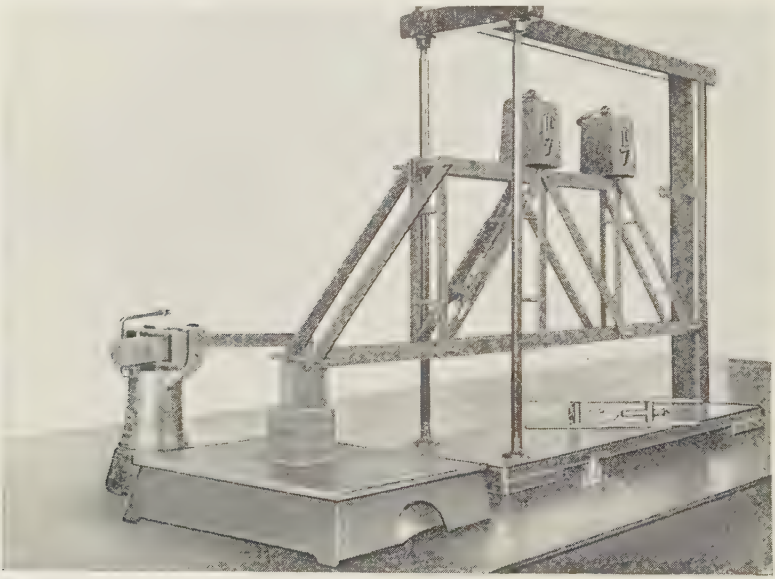


Fig. 134 A. Experimental framed beam.

In the present case a cantilever of the required length is built up, as shown in Fig. 134, of bars connected by smooth pins. If it is desired to make any "bay" rigid, this may be done either by using a solid block, see (D), Fig. 134, or by placing a diagonal bar on each side of the bay, see bays (A) and (C). The bars are made in two lengths; the shorter bars (length $= l$) may be 8 or 10 ins. long, while the longer bars (length $= \sqrt{2}l$) should be of such a length as to reach across the diagonal of a square bay. To prevent twisting, a few bars should be framed together as shown at (G), in widths varying from 2 ins. downwards.

Tensile forces are measured by a spring balance connecting two pins, as shown in bay (B). To measure compressive forces two special bars (one of length l and the other of length $\sqrt{2}l$) should be constructed as

shown at (H), in which the side bars are duplicated and can slide upon each other, while a spring balance connected to these two frames serves to measure the magnitude of any compressive force tending to bring the two ends of the frame nearer together. The complete apparatus is illustrated in the photograph in Fig. 134 A as set up for an experiment on a framed beam. The procedure to be followed will be explained in connection with each experiment.

Case I. *To find the forces acting at section* C.

For this case both the outer bay (c) and the inner bay (a) should be made rigid. See Fig. 135, Case I. At section C the cantilever should now be supported by means of the vertical spring balance S_c, and also the horizontal balance T_b. The horizontal force at the lowest point on the section may also be measured if desired; but, since we now know that the forces acting at the top and bottom will be equal in magnitude, further complication may be avoided by using a solid bottom bar.

Before adding the load W_1, adjust the balances S_c and T_b until the top bars are horizontal and take the "zero" readings. Add the load W_1 and once more adjust S_c and T_b until the top bars are horizontal and thus obtain the magnitude of the forces S_c and T_b.

It should be found that the force $S_c = W_1$; also that the force T_b is such that

$$T_b \times y = W_1 \times l_c.$$

If the load W_2 is now added, it will be found that, while the shear force at section C is increased by the amount W_2, the value of T_b, and therefore the value of the bending moment at the section, has not been affected. This follows from the fact that the load W_2 has no moment about section C.

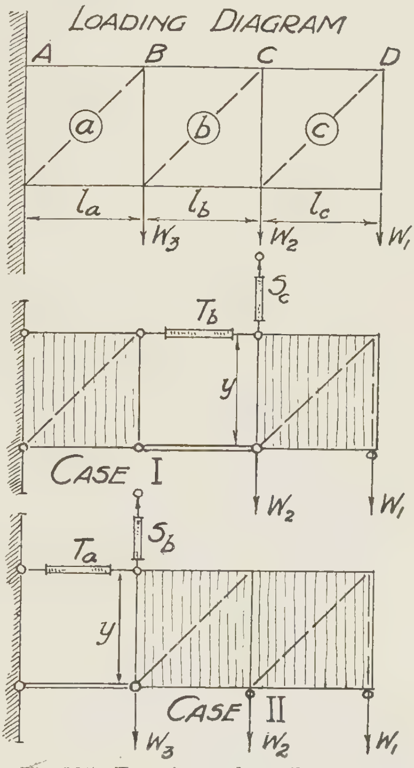


Fig. 135. Experimental cantilever with concentrated loads—Forces acting at sections B and C.

Case II. *To find the forces acting at section B.*

The apparatus should now be arranged as shown in Fig. 135, Case II, the two outer bays being made rigid and the cantilever supported at section B by the vertical spring balance S_b, the horizontal balance T_a, and a solid bottom bar, W_1 and W_2, remaining as before.

It should be found that $S_b = W_1 + W_2$, while the value of T_a is such that

$$T_a \times y = W_1 \times (l_b + l_c) + (W_2 \times l_b) = B_b.$$

If now the load W_3 is added, then, as before, S_b will be increased by this amount while the magnitude of T_a will be unaffected.

These results will be found to agree with those arrived at in the discussion given in para. 77.

80. Cantilever with uniformly distributed load. So far we have dealt only with loads concentrated at one point. Now consider a case in which the load is spread uniformly along the length of the cantilever, "uniformly distributed loading". The magnitude of such a load is usually defined by stating the number of units of weight (lbs. or tons) which are spread over each unit of length (inches or feet).

Shear Force. Assume the cantilever to be loaded as shown in Fig. 136, with w lbs. per inch of length. The shear force acting at any section C, distant l ins. from B, will be equal to the total load acting on the cantilever to the right of this section, between the points B and C. In this case

$$S_c = wl \text{ lbs.}$$

From this expression it is evident that the shear force increases directly with l, that is with the distance measured from B, so that the shear force diagram must be a triangle, the ordinates starting with zero value at b and reaching a maximum value of wL at a; see Fig. 136.

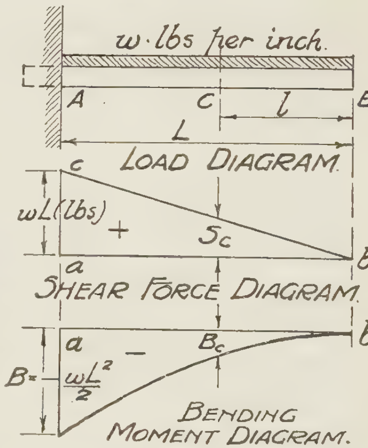


Fig. 136. Cantilever with uniformly distributed load.

Bending Moment. The bending moment at the section C, see Fig. 136, is evidently equal to the moment about this section of the total load carried by the cantilever between this point and the free end of the cantilever. But the distance between C and B is l ins. and the rate of loading is w lbs. to the inch, hence the total load acting over this length is wl lbs. As we may consider this load for our present purpose to act at its centre of gravity (C.G.), which is $l/2$ ins. from C, we have

$$B_c = wl \times l/2 = \frac{wl^2}{2} \text{ (lb. ins.).}$$

This expression shows that the magnitude of the bending moment increases as the square of the distance from the end of the cantilever. If these values be obtained and plotted for a

number of points between A and B, the ends of the ordinates will be found to lie on a curve, see Fig. 136, which is tangential to the base ab at b, where $B = 0$, and has a maximum value at a, where

$$B_{\text{max.}} = -\frac{wL^2}{2}.$$

The maximum value at a may also be expressed in terms of the total load (wL) acting on the cantilever. If we say that wL, the total load, is represented by W, then

$$B_{\text{max.}} = -\frac{WL}{2}.$$

81. Construction of a parabolic bending moment curve. The curve just obtained, by plotting the values of the bending moments at various points on the cantilever, may be shown to be part of a *parabola*, a curve which can be readily constructed by geometrical means if we are given values fixing certain points on the curve.

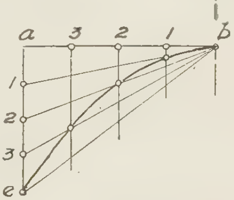


Fig. 137. Parabolic curve.

For example, in Fig. 137 let ab be the base line to which the curve is known to be tangential at b, and let ae be the maximum value for this figure or simply the value at a. Divide ab into any convenient number of equal parts and draw perpendiculars as shown. Now divide ae into the same number of equal parts and join the point b to each of the points so found. Then draw the curve, starting at b and finishing at e, so that it is tangential to the line ab at b and passes through each of the points where a sloping line cuts the correspondingly numbered vertical line, e.g. where $b1$ cuts the vertical through 1, and so on; see Fig. 137.

Note. If, for later work, the curve is to be continued to the right of b, the same process is repeated, see Fig. 138 (A), from which it will also be seen that we may consider either the line AB or the line CD to be the base line of the diagram, depending upon whether the values are set up or down from either of these lines; in any case the geometrical construction remains the same.

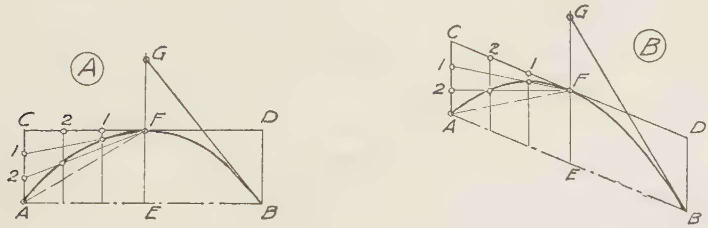


Fig. 138. Parabolic curves.

Alternative method of construction. The same geometrical method may be used where the base line AB is sloping; see Fig. 138 (B), which will sufficiently explain the construction.

If, as indicated to the right of both the cases illustrated in Fig. 138, the "centre line" EF is produced to G, so that FG is equal to FE, and

the point G joined to A and B; then the lines GA and GB are tangential to the curves at A and B in each case. This fact affords us a ready means of constructing the curve, if only the maximum value (EF) at the centre is known and a high degree of accuracy is not desired in drawing the curve.

82. Experiment. The effect of distributing the load over the length of a beam may be demonstrated experimentally by repeating the experiment described in para. 79, but replacing the concentrated loads by a series of small horseshoe-shaped weights, spaced equally along the top of the cantilever; see Fig. 139.

If, for example, W_1 is the total weight placed over the bay (c), then at section C

$$S_c = W_1,$$

and the value of T_b will be such that

$$T_b \times y = W_1 \times l_c/2.$$

These results may be compared with those obtained in para. 79, Case I. The other sections may be similarly dealt with.

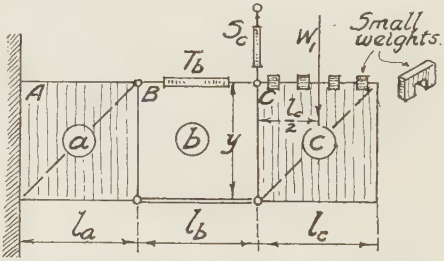


Fig. 139. Experimental cantilever with distributed load.

83. Simply supported beam with central load. See Fig. 140.

Reactions. The load W being placed centrally, each support will share equally in supporting it; hence

$$R_a \text{ (the reaction at } A) = R_b = \frac{W}{2}.$$

Shear Force. Consider the shear at any section D to the right of the centre C; the only force acting to the right of this section is R_b acting upwards, whence

$$S_d = -\frac{W}{2},$$

and this is true wherever D is taken between the points C and B.

Similarly for any section E between the points A and C; the only forces acting to right are the force R_b acting upwards

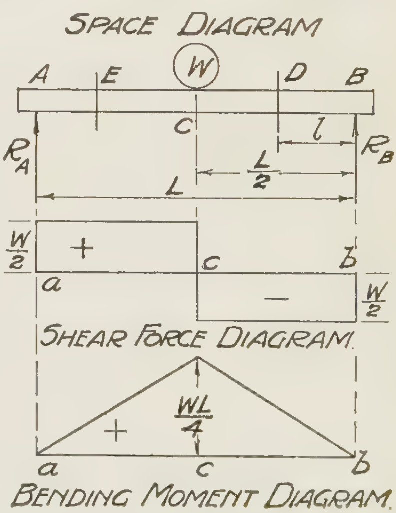


Fig. 140. Beam with central load.

and force W acting downwards, whence

$$S_e = W - \frac{W}{2} = \frac{W}{2},$$

and this is likewise true for all points between A and C.

The shear force diagram will therefore consist of two rectangular portions, as in Fig. 140, that to the left of C being positive and that to the right negative. The magnitude of the ordinates is constant throughout and equal to $W/2$.

Bending Moment. See Fig. 140. Consider first the action of the forces W and R_a about B; the moment of W about B is $W \times L/2$ acting anti-clockwise, while the moment of R_a $(= W/2)$ about B is likewise $W/2 \times L$, but acting in a clockwise direction. These two moments are equal and opposite; therefore the resultant moment at B is zero. Similarly for the resultant moment at A which is also zero.

If next we consider a section D between B and C, then the only force acting to the right of D is R_b $(= W/2)$; hence

$$B_d = W/2 \times l,$$

where l is the distance of D from B.

From this expression, which shows that B_d increases directly as the distance of D from B, it will be found that, if a number of intermediate points are taken and the values of the bending moments plotted to any convenient scale, the ends of the ordinates lie on a straight line which rises from point b. Similar considerations show that, to the left of C, the outline of the diagram is likewise a straight line rising from a. Evidently these two lines intersect over C, at which point the value of the bending moment reaches a maximum and, substituting $L/2$ for l,

$$B_{\text{max.}} = \frac{W}{2} \times \frac{L}{2} = \frac{WL}{4}.$$

84. Simply supported beam with distributed load. See Fig. 141.

Reactions. The beam is loaded at the rate of w lbs. per inch of length, the total load (W) is therefore wL lbs., where L is the length or span of the beam in inches.

Since the load is symmetrically placed, it is supported equally at A and B, whence

$$R_a = R_b = \frac{wL}{2}.$$

Shear Force. Consider the shear force at any section D, between C and B; the forces acting to the right are R_b $(= wL/2)$ acting

upwards and wl (that is the total load on the beam between the points D and B), acting downwards, whence

$$S_d = wl - \frac{wL}{2} \text{ lbs.}$$

$$= w\left(l - \frac{L}{2}\right).$$

But $L/2$ is a constant while l varies with the position of D, whence it follows that the outline of the diagram must be an inclined straight line.

At B, where l is zero, we have

$$S_b = -\frac{wL}{2}.$$

At C, where $l = L/2$, we have

$$S_c = w \times \frac{L}{2} - \frac{wL}{2} = 0.$$

Similar results will be obtained from a consideration of the shear forces acting to the left of C, except that all the values will be positive. The shear force therefore reaches a maximum over each support and passes through zero value at the centre, see Fig. 141.

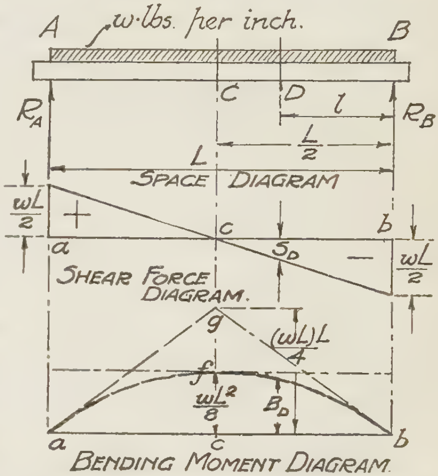


Fig. 141. Beam with uniformly distributed load.

Bending Moment. See Fig. 141. Consider the moments of the forces acting to the right of section D, the force R_b ($= wL/2$) acts at a distance l from D and its moment is therefore $wL/2 \times l$. The force due to the load wl acting on this portion may be taken to act at the C.G. of the load, that is $l/2$ from D, hence its moment is $wl \times l/2$. Giving due regard to sign, the bending moment at D is

$$B_d = \frac{wLl}{2} - \frac{wl^2}{2}$$

$$= \frac{wl}{2}(L - l).$$

The graph of this expression will give a parabolic curve. At B, since $l = 0$, $wl/2$ becomes zero and the bending moment becomes zero. Similarly at the point A, where the bending moment is also zero.

At C, where l becomes equal to $L/2$, we have, from the above expression, that

$$B_c = \frac{wL}{2} \times \frac{L}{2} - \frac{w}{2} \times \frac{L^2}{4}$$

$$= \frac{wL^2}{4} - \frac{wL^2}{8}$$

or

$$B_{\text{max.}} = \frac{wL^2}{8} = \frac{WL}{8}.$$

This is clearly the maximum value; knowing this we can complete the diagram by the method already explained in para. 81.

Effect of distribution of the load. The precise effect which distribution has upon the bending moment diagram should be noted here. Imagine the load W in Fig. 141 to have been concentrated at C, the centre of the beam, then, from para. 83, the bending moment at the centre would have been $WL/4$ (where $W = wL$), that is exactly twice the magnitude which obtains when the same load is spread over the whole length of the beam.

This fact may be used to form the basis for a method of drawing the bending moment diagram for a distributed load, the procedure being as follows; see Fig. 141:

(a) Assume the total load wL to act at the centre and draw the B.M. diagram; the maximum ordinate cg will be equal to $(wL)\,L/4$.

(b) Bisect the ordinate cg at f and also join g to a and b. The curve for the bending moment diagram for *the distributed load* will then pass through f and be tangential to the lines ga and gb at the points a and b; see also para. 81.

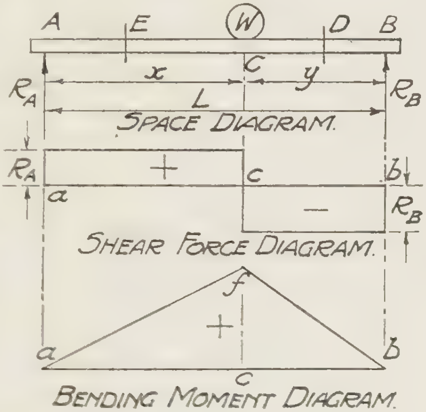


Fig. 142. Concentrated load not at the centre.

85. Concentrated load not at the centre. See Fig. 142.

Reactions. Taking moments about A we have, for equilibrium,

$$R_b \times L = W \times x,$$

$$\therefore\ R_b = \frac{Wx}{L}.$$

Similarly

$$R_a = \frac{Wy}{L},$$

or, putting these expressions into words,

Reaction (for a single load) =

$$\frac{\text{load} \times \text{distance from opposite support}}{\text{span}}. \quad \ldots\ldots\text{(i)}$$

Shear Force. As in para. 83, we have

$$S_d = R_b = -\frac{Wx}{L},$$

and this holds between points B and C. Likewise

$$S_e = R_a = \frac{Wy}{L},$$

and this holds between points A and C. The shear force diagram is therefore of the form shown in Fig. 142.

Bending Moment. The diagram of bending moment is triangular, and similar to that for a central load, the maximum ordinate occurring under the load. Considering forces to the right of C, we have

$$B_c = R_b \times y = \frac{Wx}{L} \times y,$$

or

$$B_{\text{max.}} = \frac{Wxy}{L}.$$

86. Uniformly distributed load covering a portion only of the beam. See Fig. 143. Let the load of w lbs. per inch be spread over a length of l inches. If C be the centre of this portion DE of the beam, then, so far as the reactions of the beam are concerned, we may assume the total load of wl lbs. to be acting at C, which is also the C.G. of the load.

Reactions. The reactions will therefore be

$$R_a = \frac{(wl)\,y}{L},$$

and

$$R_b = \frac{(wl)\,x}{L};$$

see para. 85.

Shear Force. For a concentrated load of wl lbs. at C the outline of the shear force diagram would therefore be the figure *aghcklb*, as shown in Fig. 143. But clearly, the effect of spreading the load uniformly over the length DE must be to change the magnitude of the shear force *uniformly*, from the positive value which it has at point D to the negative value which it has at point E, *without changing the values which it has to the left or right of this portion.* The true diagram for shear force is therefore completed by drawing the sloping line *de*.

Bending Moment. We may proceed with the bending moment diagram in a similar manner; see Fig. 143. First draw the bending

moment diagram *adceb* for a concentrated load of *wl* lbs. acting at *C*, for which

$$B_c = \left(\frac{(wl)\,x.y}{L}\right);$$

see para. 85.

It will be obvious that so long as the load *wl* lbs. continues to be spread over the portion *DE* of the beam its C.G. will remain at *C*, *and the magnitudes of the bending moments at points to the left or right of DE will remain unchanged.* It follows that the lines *ad* and *eb* form portions of the final diagram for the distributed load.

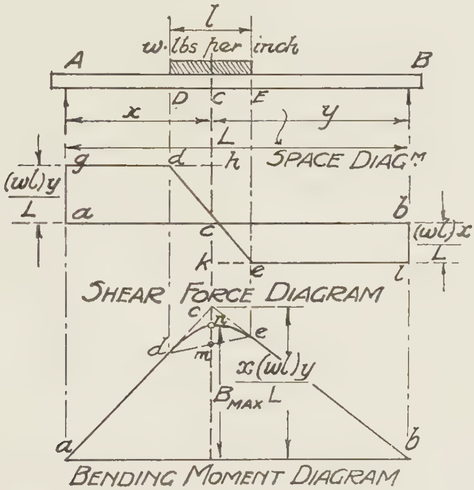


Fig. 143. Uniform load over portion of beam.

Over the portion *DE*, which for our purpose may be looked upon as a short beam of this length, the triangle *dce* evidently represents the bending moment diagram for a concentrated load of *wl* lbs. at *C*; if therefore, we bisect *cm* at *n* and draw the parabolic curve *dne* in the manner already described, then the graph *adneb* will be the correct outline of the bending moment diagram for the beam when acted upon by the load *wl* lbs. spread over the length *DE*.

If desired, mathematical expressions may be obtained for the shear force and bending moment at any point in the length of the beam, as explained in preceding paragraphs, and from these the nature of the outlines of the diagrams may be inferred and the values obtained from the above diagrams may be checked.

Example. In Fig. 143 let $x = 7$ ft., $y = 9$ ft., $l = 4$ ft. and $w = 100$ lbs. per inch. Find (a) the reactions at A and B, (b) the bending moment at F (not shown), the centre of the beam.

(*a*) The total load $= wl = 100 \times 48 = 4800$ lbs.

$$R_a = \text{reaction at } A = \frac{(wl).y}{L} = \frac{4800 \times 9}{16} = 2700 \text{ lbs.}$$

Similarly $R_b = 2100$ lbs.

(*b*) Since the point F lies within the portion DE, the next part of the problem may be solved by calculation or by using a diagram constructed in the manner explained above. Using the former method, take moments to the right of F, then

$$B_f = R_b \times 8 \text{ ft.} - (\text{moment due to weight on the portion } FE)$$
$$= 2100 \times 96 - (100 \times 12)\, 12/2$$
$$= 201{,}600 - 7200 = 194{,}400 \text{ lb. ins.}$$

87. Point Loads. So far we have spoken of loads being applied or beams supported "at points". For the purposes of calculation this is of course convenient, but it must be remembered that practical considerations compel even "concentrated" loads to be spread over short distances and beams to be supported upon fairly large "bearing surfaces". The effect of this on the shear force and bending moment diagrams is interesting and should be noted. The results obtained in the last paragraph indicate that the effect of the distribution of a "concentrated" load is to ease off the suddenness of the changes in the shear force diagram, and to reduce the maximum values in the case of the bending moment diagram; see Fig. 143.

To make the point clearer we illustrate a case in Fig. 144, in which it is first assumed that the load W is concentrated at point C and afterwards spread over the short length DE. The shear force and bending moment diagrams for both cases are given. The forces supporting the beam are then treated in the same way, being first assumed to act at the points A and B, and afterwards spread along short lengths of the beam. These diagrams should be self-explanatory.

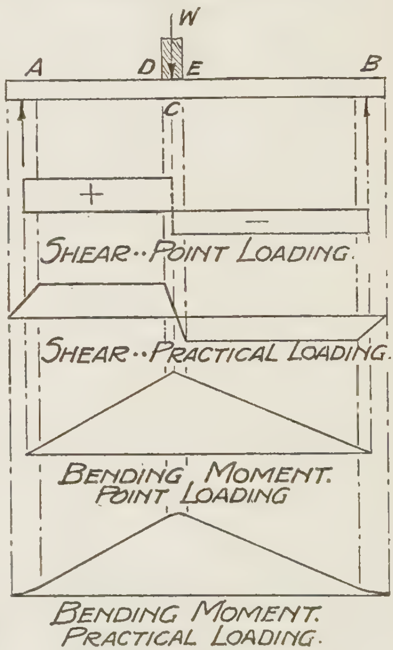


Fig. 144. Practical loading and supporting of beams.

For convenience, we will continue to assume that concentrated loads are applied at the C.G.'s of their bearing surfaces, which are usually short, and that beams and other structures are supported by forces acting at or near the centres of their supporting surfaces; such departures from actual conditions as these assumptions involve are, as we now see, on the side of safety.

88. Beam with two equal concentrated loads on symmetrical overhanging ends. See Fig. 145. This is an interesting and important case and one which should not require a lengthy explanation. The reactions R_a and R_b are clearly equal to each other and to the magnitude of the single load W. The shear values between C and A, and also between B and D, are likewise equal to W, but the shear force has a negative value in the former position and a positive value in the latter.

If we consider the forces to the right of A, we find that

$$S_a = W - R_b = W - W = 0.$$

Hence *The shear force between* A *and* B *is zero* ......(a).

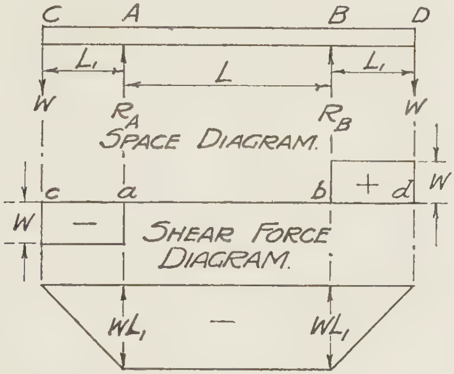


Fig. 145. Beam with concentrated loads on overhanging ends.

Now consider the moment of the forces acting to the right of B, from which we find

$$B_b = -WL_1.$$

Similarly, from a consideration of the forces acting to the right of A,

$$B_a = R_b \times L - W(L + L_1) = WL - WL - WL_1$$
$$= -WL_1.$$

These results could also have been deduced from the fact that the forces acting at B and D constitute a clockwise couple of magnitude WL_1, while the forces acting at C and A likewise constitute a couple of the same magnitude but of opposite sign. It thus follows that

Between the points A *and* B *the beam is subjected to a constant negative bending moment equal to* $\mathrm{WL_1}$ ......(b).

89. Beam with overhanging ends and a uniformly distributed load. See Fig. 146. In dealing with this case it will be a convenience to consider the load in two portions, viz. that covering the centre portion and that covering the two outer portions. The final diagrams will then be obtained by adding together the two portions previously worked out.

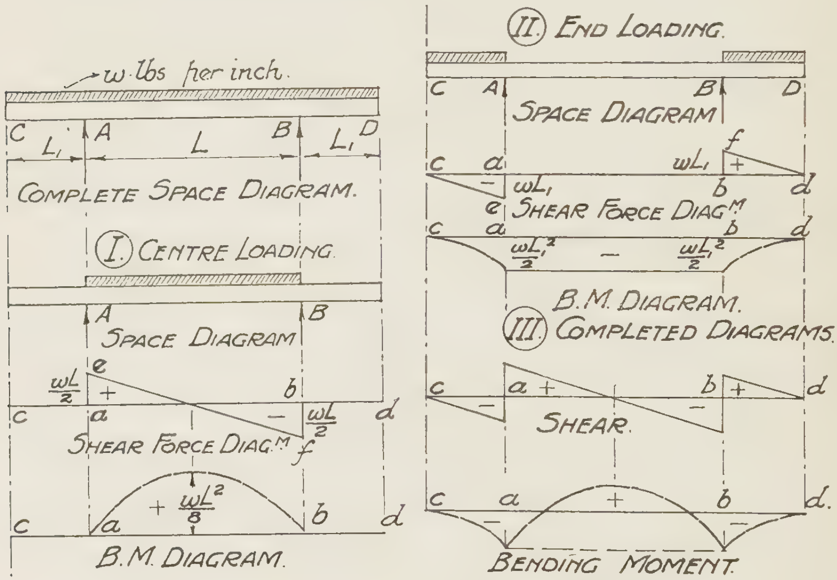


Fig. 146. Uniform load on beam with overhanging ends.

I. Centre loading. See Fig. 146 (I). The total load on this portion of the beam will be wL lbs. This will be borne equally by the two supports and hence the shear at A and at B will be $wL/2$, being positive at the former point and negative at the latter. Since the shear values alter uniformly from one to the other value, the diagram will be completed by drawing the sloping line ef. The bending moment diagram will be of the same form as for a simply supported beam with distributed loading, the maximum ordinate being equal to $wL^2/8$; see Fig. 146 (I).

II. End loading. See Fig. 146 (II). The shear values at A and B will clearly be each equal to wL_1 but of opposite sign. The shear value between A and B will be zero, while beyond A and B it will decrease uniformly to zero at the extreme ends.

The bending moment diagrams for the outer ends will be of the same form as for a uniformly loaded cantilever, and for this case, as

in that dealt with in the preceding paragraph, the bending moment will remain constant between the points A and B, the magnitude being $-wL_1^2/2$.

III. Completed diagrams. See Fig. 146 (III). The diagrams of shearing force will be easily added together, each section being unaffected by the other. In the case of the bending moment diagrams, however, when one portion is superimposed on the other we see that certain portions coincide: then, since these portions are marked positive on one diagram and negative on the other, they obviously "cancel out" and the uncancelled portions form the completed bending moment diagram shown. It should be noted that in this case negative bending moments occur at the supporting points A and B, while the bending moment at the centre of the beam is positive.

90. Beam subjected to several concentrated loads. See Fig. 147. This merely requires an extension of the methods already described, the separate diagrams for each load being added to obtain the complete and final figure.

Reactions. Each load is considered separately, as shown below, but can be expressed in one equation. Thus,

$$R_a \text{ due to } W_1 = \frac{W_1 \times x}{L}$$

$$R_a \text{ due to } W_2 = \frac{W_2 \times y}{L},$$

$$R_a \text{ due to } W_3 = \frac{W_3 \times z}{L},$$

or, taking these together,

$$\textbf{Total reaction at } \boldsymbol{A} = R_a = \frac{W_1 x}{L} + \frac{W_2 y}{L} + \frac{W_3 z}{L}$$

$$= \frac{W_1 x + W_2 y + W_3 z}{L}.$$

The reaction at B will be equal to the total load less the reaction at A, or,

$$R_b = (W_1 + W_2 + W_3) - R_a,$$

but, as a check, it should always be calculated separately.

Shear Force Diagram. If the total positive shear force is set up at A and the negative shear force is set down at B, the shear diagram can be completed by stepping down (or up), at each point at which a load is applied, by the amount of that load; see Fig. 147.

Bending Moment Diagram. The small triangular figures $a1b$, $a2b$ and $a3b$ represent the bending moment diagrams for the loads W_1, W_2 and W_3 respectively. The ordinates so obtained are then added together, to give the complete bending moment diagram *acdeb*; see Fig. 147.

If it is desired to check the values, or to find these values directly by calculation, a complete expression can be set out as follows. Taking the forces acting to the right of C, we have

$$B_c = R_b x - W_2(x-y) - W_3(x-z):$$

and similarly at other sections.

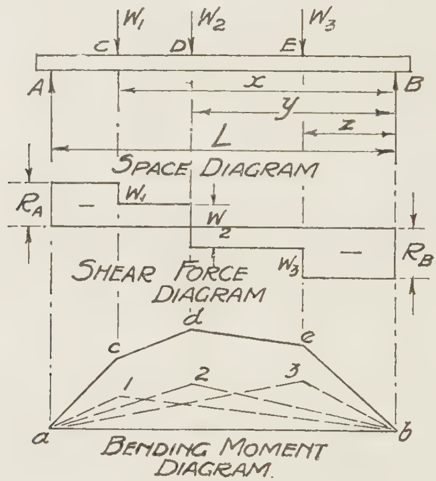


Fig. 147. Beam with several concentrated loads.

91. Graphical method of construction for the bending moment diagram. In examples such as is shown in Fig. 147, where the beam is subjected to a series of isolated loads, it is useful, as a check, to draw the shear and bending moment diagrams by graphical methods, using the link and polar diagrams described in Chap. IV. (The method may also be applied to continuous loads by first dividing the load into small sections and treating each as a separate load.) The procedure is not difficult, but special care must be taken to see that the scale for the bending moment diagram is correctly calculated; the method and its proof are given below.

Let a beam AB be loaded as shown in Fig. 148. Let us assume that the beam is drawn to a linear scale of 1 in. to X ins. Adopting a suitable load or force scale of, say, 1 in. to Y lbs., set down the load line *1234*

in the manner described in Chap. IV. Selecting any convenient pole P, complete the polar diagram.

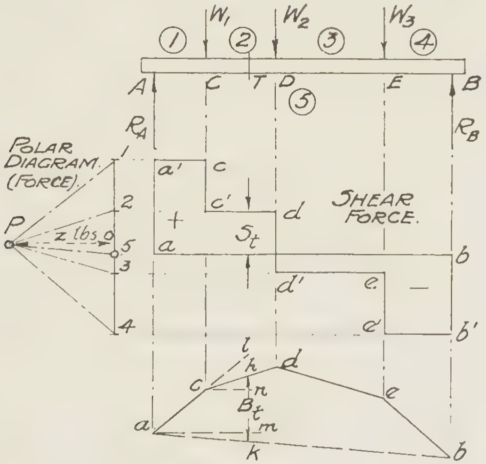
(The relative positions of the diagrams shown in Fig. 148 will be found to be convenient and should be followed in solving other problems.)

Reactions. In the manner already described draw the links *ac*, *cd*, *de* and *eb* of the link polygon. Complete the figure by drawing the "closing link" *ab* and draw *P5*, through P, parallel to this line. Evidently the lines *45* and *51* represent the magnitudes of the reactions at B and A respectively.

Shear Force Diagram. The two reactions may now be projected across, in the manner shown, upon the vertical lines in which these forces are acting and on which they are then represented by bb' and aa' respectively. The line *ab* forms the base line of the shear force diagram. Between the points A and C, and also points B and E, on the beam the values of the shear force do not alter; hence these portions of the diagram can be completed by drawing short horizontal lines.

At C, where the load W_1 is added, the value of the positive shear force will be reduced by an amount equal to W_1. Similarly at point D. These reductions are made by direct projection horizontally from the points already found on the load line *12534*. Between C and D, and again between D and E, the values remain constant and the diagram is completed by drawing the horizontal lines $c'd$ and $d'e$ respectively.

The shear force at any point T on the beam is given by the corresponding ordinate (S_t) on the diagram, the scale being that adopted for the load line.



Linear scale, 1 in. to X ins.; Load scale, 1 in. to Y lbs.; $PO = Z$ lbs.;
Bending moment scale, 1 in. to $(X \times Z)$ lb. ins. (See text.)

Fig. 148. Graphical method of drawing shear force and bending moment diagrams.

Bending Moment Diagram. We will now proceed to show that *the link polygon* acdeb *is also a bending moment diagram*, the vertical ordinates of which are proportional to the bending moment at corresponding points on the beam.

Let hk be the ordinate intercepted by the links of the link polygon on the vertical line passing through any section T on the beam. Produce the link ac to meet this vertical line in l. Also draw the *horizontal* lines am and cn through a and c respectively, cutting this line in m and n. On the polar diagram draw the *horizontal* line PO through P cutting the load line in O.

Consider first the bending moment at T due to the reaction R_a at A. This is equal to $R_a \times AT$. But the distances AT and am are equal, while the force R_a is represented on the load polygon by the distance 1.5. Hence we have

$$\text{Bending moment at } T \text{ due to } R_a = R_a \times AT = 1.5 \times am. \quad \text{......(a)}$$

Again, since in the triangles alk and $P.1.5$ corresponding sides are parallel, these two triangles are similar. Also, since the lines am and PO are parallel and similarly placed in these triangles, they must bear the same ratio to each as do the sides of the triangle, hence

$$\frac{PO}{am} = \frac{1.5}{lk}, \text{ or } lk \times PO = 1.5 \times am.$$

But the right-hand side of this expression is equal to $R_a \times AT$, see (a) above; therefore

$$lk \times PO = \text{bending moment of } R_a \text{ about } T. \quad \text{......(b)}$$

Similarly we may show that

$$lh \times PO = \text{bending moment of } W_1 \text{ about } T. \quad \text{......(c)}$$

Now the moment of R_a about T is positive, while the moment of W_1 about T is negative. The total bending moment (B_t) at the section T will therefore be equal to the difference between these two magnitudes, hence from (b) and (c)

$$\begin{aligned}\textbf{Total moment at } \boldsymbol{T} = \boldsymbol{B_t} &= (lk \times PO) - (lh \times PO) \\ &= PO\,(lk - lh) = \boldsymbol{PO \times kh}; \quad \text{......(d)}\end{aligned}$$

see Fig. 148.

Now if the statement in (d) be true for the forces acting to the left of T, it must likewise be true for the forces acting to the right of T, since we obtain the same result for the bending moment at T whether we consider the forces acting to the right or to the left of that section. In addition, it can be shown to be true no matter where the section T is chosen in the length of the beam. It follows that the diagram $acdeb$ *is a bending moment diagram*, and its ordinates, when measured to some scale, will give the magnitude of the bending moment at each and every section along the beam.

Bending Moment Scale. From the expression (d) we have

$$\text{Bending moment at } T = B_t = PO \times kh.$$

Now PO represents the *horizontal component* of any and every inclined force acting in the polar diagram, see Fig. 148, and is therefore a constant.

Again since, as we have seen (see Chap. IV), the link polygon $acdeb$—which we are now using as a bending moment diagram—is really a "space diagram", the ordinate kh is a "length" and will be measured on the linear scale.

We may therefore re-write the expression (d) as

$$\begin{aligned}\text{Bending moment at } T = B_t = PO &\text{ (measured on the force scale)} \\ \times\, hk &\text{ (measured on the linear scale).}\end{aligned}$$

For example, let the *linear scale* in Fig. 148 be 1 in. to 12 ft., and let PO represent 5000 lbs. on the *force scale.* If then the ordinate hk at T actually measures $\frac{3}{4}$ in., this will represent 9 ft. on the *linear scale*, and the bending moment at $T = B_t = 5000$ lbs. $\times$ 9 ft. $= 45{,}000$ lb. ft.

But if $\frac{3}{4}$ in. represents 45,000 lb. ft. then 1 in. will represent 60,000 lb. ft., and this is evidently the *bending moment scale* in this case.

Dividing 60,000 lb. ft. by 5000 lbs., the magnitude of force PO, we get 12 ft., which is the number of feet represented by 1 in. on the *linear scale.* Hence we have the following simple rule: *To find the bending moment scale multiply the linear scale by* PO, *the constant horizontal component of the forces acting in polar diagram.*

In order to give a convenient scale the distance PO should be carefully chosen. In Fig. 148, where the *linear scale* is 1 in. to X ins., and PO represents Z lbs., the *bending moment scale* would be given by 1 in. to $(X \times Z)$ lb. ins.; both X and Z might conveniently be in multiples of tens, hundreds or thousands according to the magnitudes of the quantities to be dealt with.

Problems IX

1. If in Fig. 131 the length of the cantilever is 8 ft. and the load (W) is 4 tons, what will be the shear force (in lbs.) and bending moment (in lb. ins.) at a point C, 3 ft. 6 ins. from the end B of the cantilever?

2. If in Fig. 136 the value of w is 100 lbs. per inch, find the shear force and bending moment at a point C, 80 ins. from the end B.

3. If in Fig. 140 the span L is 20 ft. and W is 5000 lbs., find the maximum shear force and bending moment.

4. If in Fig. 141 L is 15 ft. and w is 2000 lbs. per foot, what is the maximum bending moment? What is the shear force and bending moment at a point 5 ft. from B?

5. If in Fig. 142 the span L is 20 ft. and the distance y is 8 ft., what is the maximum bending moment when W is 5 tons?

6. Find, by drawing and measuring the bending moment diagram, the magnitude of the maximum bending moment in Fig. 143 if the span is 20 ft. and y is 14 ft., l is 4 ft. and w is 2000 lbs. per ft.?

7. If in Fig. 146 L is 12 ft. while L_1 is 5 ft. and w is 200 lbs. per inch, find the bending moments at each support and at the centre of the beam.

8. With a beam of the same length as in Prob. 7, and with uniformly distributed loading over the whole length, where would the supports A and B have to be placed so that the bending moments at these points would be of equal magnitude to the bending moment at the centre of the beam? (Note. Write down the bending moments at these points in general terms and obtain L_1 in terms of L.)

9. If in Fig. 147 W_1 is 5 tons, W_2 is 3 tons, W_3 is 4 tons, L is 22 ft. (x being 18 ft., y 12 ft. and z 6 ft.), find the reactions at A and B and the total bending moments at C, D and E.

10. If in Fig. 148 PO measures 5 ins., the Force Scale is 2000 lbs. to an inch and the Linear Scale is $\frac{1}{2}$ inch to a foot, what is the Bending Moment Scale?

CHAPTER X

RELATIONS BETWEEN LOAD, SHEAR FORCE AND BENDING MOMENT IN A BEAM

92. Relations between the Shear Force Diagram and the Bending Moment Diagram.

A. Particular case—Isolated loads. Let us examine the case of a beam loaded as shown in Fig. 149.

Reactions. Proceeding as already explained, we have:

$$\text{Reaction at } A = \frac{5 \times 16}{20} + \frac{3 \times 5}{20}$$

$$= \frac{80 + 15}{20},$$

or $$R_a = 4\tfrac{3}{4} \text{ tons.}$$

Similarly $$R_b = \frac{3 \times 15}{20} + \frac{5 \times 4}{20}$$

$$= 3\tfrac{1}{4} \text{ tons.}$$

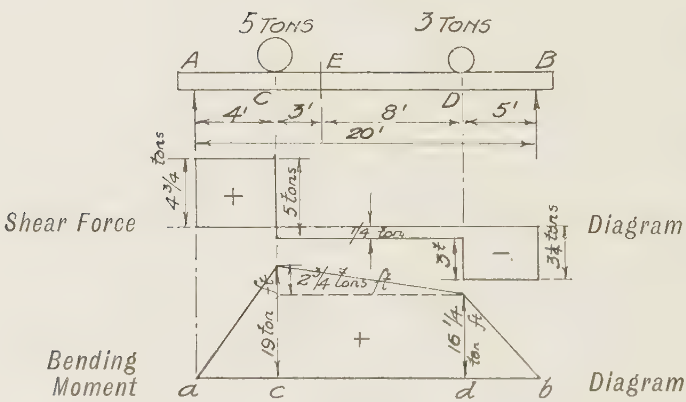


Fig. 149.

Shear Force. From these figures we may set out the shear diagram as shown.

Bending Moment. At any section, say E, 8 ft. from D, take moments on the right; then

$$\text{Bending moment at } E = \tfrac{13}{4} \times 13 - 3 \times 8,$$

or $$B_e = 18\tfrac{1}{4} \text{ ton feet.} \qquad \ldots\ldots(a)$$

Now consider the **area** of the shear force diagram up to section E, expressing this "area" in terms of tons (vertical ordinates), and feet (horizontal distances). Then the area of the shear diagram from the left-hand free end up to E

$$= (4\tfrac{3}{4} \times 4) \text{ ton ft.} - (\tfrac{1}{4} \times 3) \text{ ton ft.}$$

$$= 18\tfrac{1}{4} \text{ ton ft.} \qquad \text{......(b)}$$

Similarly, the area of the shear diagram from the right-hand free end up to E

$$= -(\tfrac{1}{4} \times 8) - (3\tfrac{1}{4} \times 5)$$

$$= -18\tfrac{1}{4} \text{ ton ft.} \qquad \text{......(c)}$$

Comparing (a), (b) and (c) we see that the *magnitudes* so obtained are identical.

This relation will be found to exist no matter where E is chosen. Similarly, it will be found to be true of any of the examples which we have already dealt with or may deal with later. We may therefore state that:

I. The magnitude of the bending moment at any section of a beam is equal to the area of the shear force diagram between that section and a free end of the beam.

B. General case—Isolated loads. Let us now consider a more general case such as that illustrated in Fig. 150, which represents a portion of a beam between two sections A and B a distance x apart, loads W_1 and W_2 being applied at A and B respectively. The magnitude of the shear force (S_a) and the bending moment (B_a), at the section A, being indicated on the corresponding portions of the shear force and bending moment diagrams.

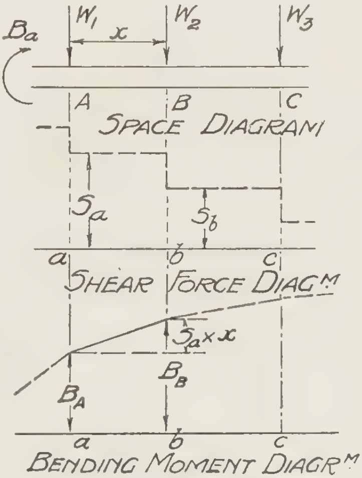


Fig. 150.

Now B_a represents the moment of all the forces to the left of A, while S_a represents the sum of all these forces up to and including the force W_1, and it is possible to show that the total bending moment at B will be equal to the bending moment at A, *plus* the moment about B of a force equal in magnitude to the shear force S_a, or, $B_b = B_a + (S_a \times x)$. In other words, the *increase* in the magnitude of the bending moment at B over that at A is $(S_a \times x)$. It will be seen that the quantity

$(S_a \times x)$ is equal to the area of the shear force diagram between the two points A and B.

Rate of change of Bending Moment. Since the change just ascertained is spread over the distance x between A and B, the **rate** at which the magnitude of the bending moment is changing between the two points will be given by the expression

$$\textbf{rate of change} = \frac{\text{total change}}{\text{distance}},$$

which in this case $$= \frac{S_a \times x}{x},$$

which **equals S_a, the shear force at A,** ......(d)

An actual case will make this clearer.

Example. In the case illustrated in Fig. 149 the bending moment at C is 19 ton ft., while at D it is $16\frac{1}{4}$ ton ft.

Then, to check the statement made above, we may write

$$\begin{aligned}\text{Bending moment at } D &= (\text{bending moment at } C) + (S_c \times \text{distance } DC)\\ &= 19 \text{ ton ft.} + (-\tfrac{1}{4} \text{ ton} \times 11 \text{ ft.}), \text{ see Fig. 149,}\\ &= 19 - 2\tfrac{3}{4} = 16\tfrac{1}{4} \text{ ton ft.}\end{aligned}$$

as before, which confirms the statement.

The change of magnitude in the bending moment between C and D is $-2\frac{3}{4}$ ton ft., and this takes place over a distance of 11 ft. The rate of change is therefore $-(2\frac{3}{4} \div 11)$, or $-\frac{1}{4}$ ton, and this, as we see, is equal to the shear force.

C. General case—Distributed loading. If we examine the case illustrated in Fig. 150, we see that under the conditions indicated the relation between the rate of change of bending moment and the magnitude of the shear force is unaffected by the magnitude of the distance x. If then the loading on this portion of the beam ABC is a uniformly distributed load, we can assume it divided up into a large number of equal parts, of which the loads W_1, W_2 and W_3 may represent consecutive portions. Over this section the outline of the bending moment diagram would then be made up of a number of short straight lines, which would approximate to a continuous curve, since the change from the slope of one line to that of the one next to it would be regular.

If we took a very large number of these equal portions, that is to say if we made x extremely small, this outline curve would, as we already know, be a parabolic curve in the case of a uniformly distributed load.

The *rate* at which the bending moment ordinates are increasing (or decreasing) may evidently be measured by the *inclination* or *slope* of the outline of the bending moment diagram, and, as will

be understood from a knowledge of geometry, the slope of a curved line at any point is given by the slope of the tangent drawn at that point. There need therefore be no difficulty about ascertaining the changes of slope at successive points on a curve. Hence, remembering that the points A and B on the outline of the bending moment diagram in Fig. 150 can be taken so close together that they practically become one and the same point, we may put the statement (d) above into a general form, *which is independent of the exact manner of loading*, as follows:

II. The rate of change of the bending moment at any section of a beam is given by the shear force at that section.

D. Position of maximum bending moment. It follows from statement II that, when the shear force at a section is zero, then the rate of change of bending moment at that section is likewise zero. In such a case the outline of the bending moment diagram will be neither rising nor falling, that is, it will be horizontal or, more correctly, it will be parallel to the base line of the bending moment diagram. For example, in Fig. 145 the shear force has zero value between the points A and B on the beam and, between the same points, the outline of the bending moment diagram is horizontal.

Let us, however, examine a case in which the shear values pass through a zero value at a point. In Fig. 151 such a case is illustrated, the shear force being zero at the point f. Following the outline of the bending moment diagram from the end A, since the shear force remains constant and is positive between the sections A and C, therefore the rate of change of bending moment is likewise constant, and the outline of the bending moment diagram is a straight line inclined *upwards* to the right.

From C the value of the shear force falls off uniformly to the section D, hence the rate of change of the bending moment, that is the rate of change of slope of the outline of the diagram, will also fall off uniformly, so that between c' and d' the outline of the bending moment diagram must be a regular curve; see also para. 86.

Now up to the point f on the shear force diagram the shear force is positive, so that the magnitudes of the bending moments, which are given by the area of the shear force diagram, must go on increasing up to this point f. As we pass beyond point f the shear force has a negative value, so that from f onwards to the right the total sum of the area of the shear force diagram, that is the magnitude of the bending moments, must decrease. It is therefore clear that *at section* F *the bending moment reaches a maximum value*, point f' being the point at which the outline of the bending moment diagram is horizontal. This can be shown

to be so by drawing a line parallel to the base line *ab* of the diagram, this line will touch or be tangential to the curve at f', the position of maximum bending moment.

From the above analysis, it is clear that the value of the bending moment will reach a maximum when the shear force has zero value, that is wherever the outline of the shear force diagram crosses the base line of the diagram. In certain cases, such as that shown in Fig. 146, there may be several such points in the length of a beam.

We may now put the above statement in more general form as follows:

III. Maximum bending moments in a beam occur at those sections at which the magnitude of the shear force is zero.

In designing beams it is necessary that we should know both the positions and the magnitudes of the maximum bending moments. As the following worked example will show, we can do this without much trouble by utilising the statements set out in this chapter.

Example. *The beam shown in Fig. 151 carries a uniformly distributed load over a portion of its length. Find the magnitude and position of maximum bending moment.*

In this case the loading is at the rate of 2 tons per foot of length from C to D, a distance of 6 ft. The total load thus imposed is therefore (6×2) or 12 tons. We can look upon this as acting at E, the C.G. of the load, from which we have

$$R_a = \frac{12 \times 15}{21} = \frac{60}{7} = 8\tfrac{4}{7} \text{ tons.}$$

Similarly the reaction at B is

$$R_b = 3\tfrac{3}{7} \text{ tons.}$$

The bending moment at C is given by

$$B_c = \tfrac{60}{7} \times 3 = \tfrac{180}{7} = 25\tfrac{5}{7} \text{ ton ft.}$$

Similarly

$$B_d = \tfrac{24}{7} \times 12 = \tfrac{288}{7} = 41\tfrac{1}{7} \text{ ton ft.}$$

Solution by drawing. With the above values both the shear force diagram and the bending moment diagram can be completed, in the manner already explained in para. 86.

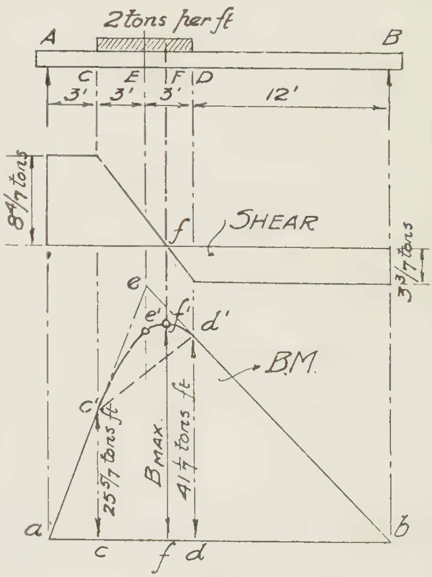


Fig. 151.

The outline of the bending moment diagram is $ac'e'd'b$; if this has been carefully drawn the magnitude of the maximum bending moment, which is given by the ordinate ff', may be obtained

with considerable accuracy by measurement. In any case it is perhaps advisable, as an additional check, to calculate this value and to find the exact position at which it occurs.

Solution by calculation. As we have shown, the maximum bending moment will be found at the section F over the point f on the shear diagram, at which the shear value is zero. Now the shear force at C is $8\frac{4}{7}$ tons. From that point onwards to D it decreases at the rate of 2 tons per foot. It therefore reaches zero value at F, which is $(\frac{60}{7} \div 2)$ or $4\frac{2}{7}$ ft. from C. Then F is $7\frac{2}{7}$ ft. from A.

The maximum bending moment, which occurs at F, is therefore

$$\begin{aligned} B_f &= (8\tfrac{4}{7} \times 7\tfrac{2}{7}) - (4\tfrac{2}{7} \times 2)\,(4\tfrac{2}{7} \times \tfrac{1}{2}) \text{ ton ft.} \\ &= 8\tfrac{4}{7}\,(7\tfrac{2}{7} - 2\tfrac{1}{7}) = 8\tfrac{4}{7} \times 5\tfrac{1}{7} \\ &= 44\tfrac{4}{49} \text{ ton ft.} \end{aligned}$$

93. Beams carrying loads which increase at a uniform rate from one end to the other. This form of loading, which is not so unusual as might at first appear, occurs when the gable-end of a lean-to building is carried by a beam, see Fig. 152 (A); when a floor

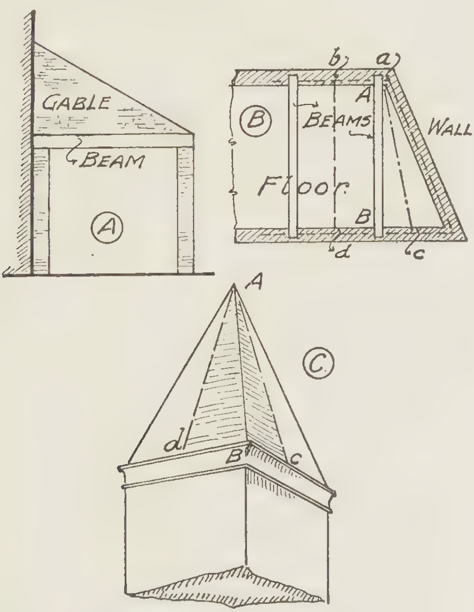


Fig. 152.

finishes against a wall which is not at right angles to the length of the floor, see Fig. 152 (B), where the load-area carried by the beam AB is enclosed in the figure $acdb$; when a pyramidal

form of roof is supported on the corner by a hip or angle rafter, see Fig. 152 (C); or when the side of a tank, subjected to water pressure, may be considered to be supported at the top and bottom edges. The same form of loading may occur on cantilevers, including the case of a reinforced concrete retaining wall and the side of a water tank unsupported at the upper edge.

It will be seen that these cases are of general importance. They also afford interesting applications of the relations between bending moment and shear force which we have discussed in this chapter. We will discuss the two principal cases, the cantilever and the beam, in general terms.

Case I. *A cantilever carrying a load which increases uniformly from the free end.* See Fig. 153.

Shear Force. Starting at B, where the load is zero, let it be assumed that the loading increases at the rate of w lbs. per inch of length as we approach A. Then at C, which is, say, a distance x from B, the intensity of the loading will be wx lbs., and the total load on this length will be

$$\left(wx \times \frac{x}{2}\right) \text{ or } \frac{wx^2}{2}. \quad \ldots\ldots(a)$$

Similarly the intensity of loading at A will be wL lbs., while the total load over the whole length of the cantilever will be

$$\left(wL \times \frac{L}{2}\right) \text{ or } \frac{wL^2}{2} \text{ lbs.} \quad \ldots\ldots(b)$$

This value (b) evidently gives the maximum shear force at A, just as the smaller value (a) gives us the shear force at C, or

$$S_c = \frac{wx^2}{2}. \quad \ldots\ldots(c)$$

It is clear from (c) that the ordinates of the shear diagram vary as the squares of their distances from B; the curve is therefore parabolic, being tangential to the base line at b and reaching its maximum value at a where

$$S_a = wL^2/2.$$

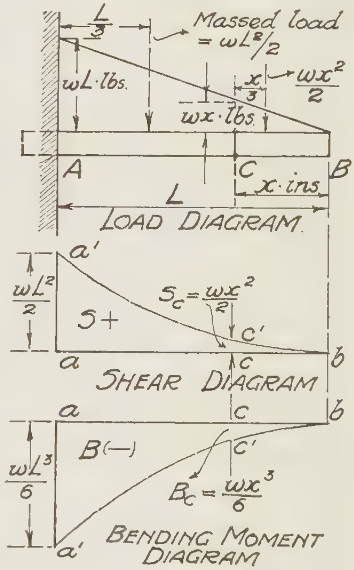


Fig. 153. Cantilever with uniformly increasing load.

Bending Moment. It will be interesting to obtain the bending moment diagram in two ways. (1) Proceeding from first principles, we see that the load on the portion BC is given by the small triangle in the load diagram. The total weight of this is $wx^2/2$ lbs. and this may be taken to act at the c.g. of the triangle, or x 3 ins. away from C. Hence we have

$$B_c = \frac{wx^2}{2} \times \frac{x}{3} = \frac{wx^3}{6} \text{ lb. ins.} \quad \ldots\ldots(d)$$

From (d) it is clear that the ordinates of the bending moment diagram vary as the cubes of their distances from B; this gives a curve which is part of a "cubic" parabola and which is tangential to the base line at b and reaches its maximum value at a, at which, since x becomes L, we have

$$B_{\text{max.}} = \frac{wL^3}{6} \text{ lb. ins.} \qquad \text{......(e)}$$

(2) Let us now proceed to find the ordinates of the bending moment diagram by means of the methods outlined in this chapter. The bending moment at C will be given by the area of the shear diagram up to that section, that is by the area bcc'.

Now the small rectangle, of base bc and height cc', which encloses this figure is divided into two portions by the outline of the shear diagram between b and c', which we know to be part of a parabolic curve. By the rules of mensuration we know that this curve divides the rectangle into two portions, the larger of which is just twice the area of the smaller, or the area of the small figure bcc' is one-third of the containing rectangle. Therefore

$$\begin{aligned} B_c &= \text{bending moment at } C \\ &= \text{area of shear diagram up to } C \\ &= \left(x \times \frac{wx^2}{2} \times \frac{1}{3}\right) \\ &= \frac{wx^3}{6}, \end{aligned}$$

which is the same value as in (d) above.

Similarly, the maximum value of the bending moment ($B_{\text{max.}}$) occurs at A, and is equal to the total area of the shear diagram, or

$$\begin{aligned} B_{\text{max.}} &= \frac{wL^2}{2} \times L \times \frac{1}{3} \\ &= \frac{wL^3}{6}, \text{ as in (e) above.} \end{aligned}$$

Case II. *A beam carrying a load which increases uniformly from one end to the other.* See Fig. 154.

Reactions. We will use the same nomenclature as in the last example. As before, the total load will be $wL^2/2$. This can be taken to act at the C.G. of the load diagram at a distance of $L/3$ from A. Hence the reaction at A is given by

$$R_a = \frac{wL^2}{2} \times \frac{2}{3} L \times \frac{1}{L} = \frac{wL^2}{3}.$$

Similarly the reaction at B will be

$$\begin{aligned} R_b &= \frac{wL^2}{2} \times \frac{L}{3} \times \frac{1}{L} \\ &= \frac{wL^2}{6}. \end{aligned}$$

These values give the magnitude of the shear force at each end of the beam, that at A being positive, while that at B is negative.

Shear Force. Consider the shear force at section C, giving due regard to sign. This will be

$$S_c = -\frac{wL^2}{6} + \frac{wx^2}{2}. \qquad \text{......(a)}$$

This expression varies as the square of the distance from B, hence the curve will be parabolic as in the last example. The curve passes through a zero value which we will presently ascertain, and then continues until, at A, it reaches the maximum value of $wL^2/3$.

Bending Moment. Now consider bending moment at C. This will be equal to the sum of the moment of the reaction at B about C and the moment of the weight of the portion of the load between C and B about C. Hence we have

$$B_c = \frac{wL^2}{6} \times x - \frac{wx^2}{2} \times \frac{x}{3} = \frac{wL^2x}{6} - \frac{wx^3}{6}. \qquad \text{......(b)}$$

This expression shows that the value of the bending moment will vary as the cube of the distance from C, hence the curve will be that of a cubic parabola as before. Since the beam is freely supported, the end bending moments will in each case be zero, so that the bending moment diagram may be completed by joining the points a and b.

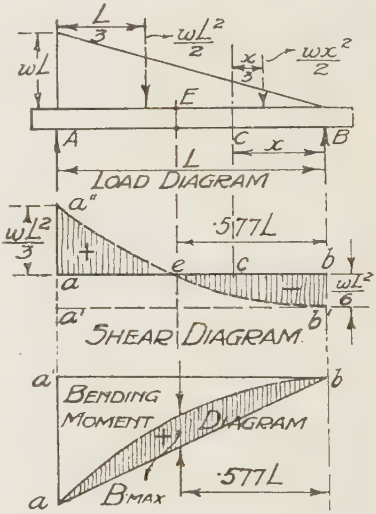


Fig. 154. Beam with uniformly increasing load.

Maximum Bending Moment. From what we have done already, we know that the maximum bending moment will occur at the section where the shear force is zero; if therefore we put S equal to zero in expression (a), we shall obtain the position of point e at which the outline of the shear force diagram crosses the base line.

$$S_e = 0 = -\frac{wL^2}{6} + \frac{wx^2}{2},$$

or $$\frac{wx^2}{2} = \frac{wL^2}{6},$$

or $$x^2 = \frac{2wL^2}{6w},$$

whence $$x = \frac{L}{\sqrt{3}} = 0{\cdot}577L.$$

Substituting this value of x in the expression (b), we have

$$\begin{aligned} B_{\text{max.}} &= \frac{wL^2}{6}(0{\cdot}577L) - \frac{w}{6}(0{\cdot}577L)^3 \\ &= \frac{0{\cdot}577wL^3}{6}(1 - 0{\cdot}577^2) \\ &= 0{\cdot}096wL^3 \times 0{\cdot}668 \\ &= 0{\cdot}064wL^3. \end{aligned}$$

94. Sum Curves. If the load, shear and bending moment diagrams in the two preceding examples, Figs. 153 and 154, are examined, it will be seen that in each diagram the ordinate of the curve at any section represents the area of the preceding diagram up to that

section. Curves so related are known as **sum or integral curves** (see any text-book on mathematics). It may be shown, both graphically and mathematically, that in all cases the outline of the shear diagram is the sum curve of the load diagram, while the outline of the bending moment diagram is likewise the sum curve of the shear diagram.

If in the two examples just completed the same values had been given to L and w in each case, not only would the relations between the curves have been as stated, but the curves would actually have been identical. The completed diagrams would then have differed only in the positions of the base lines. In Case II, for example, the position of the base line in the shear force diagram is settled by the known reactions at each end. In the case of the bending moment diagram it is settled by the fact that the bending moment is zero at each end, and therefore the base line merely joins the ends of the curve. Occasionally this affords a convenient method for constructing the bending moment diagram; see also Chap. XVII.

Problems X

1. If the rate of loading over CD in Fig. 151 be 8 tons per ft., what would be the maximum bending moment and where would it occur?

2. If the cantilever AB in Fig. 153 carries a brick gable 9 ins. thick, weighing 120 lbs. per cu. ft., state the rate of increase of loading per foot if the span (L) is 15 ft. and the height of the gable at A is 5 ft. Draw the shear force and bending moment diagrams and indicate the maximum shear force and bending moment.

3. If the gable described in Prob. 2 is carried by a beam over the same span, ascertain the positions of the ends of the new base lines so that you may complete the shear force and bending moment diagrams by using the curves obtained in Prob. 2. What is the value and position of the maximum bending moment?

CHAPTER XI

SHEAR FORCES AND BENDING MOMENTS IN FRAMED STRUCTURES

95. Framed structures subject to Bending Moment and Shear Force. All framed structures which are used to span openings may be looked upon as beams. We should therefore expect to find that there exist definite relations between the forces acting in the members of such structures, and the shear forces and bending moments which may act upon them from point to point in their length. These relations are numerically very simple in the case of framed structures, such as girders, having parallel top and bottom members; see Figs. 155–158. To deal with such cases we shall develop in this chapter a method—known in its more general form as the "method of sections" or "Ritter's method"—of ascertaining by calculation the forces acting in the members. In some cases the method is easier to apply than the graphical methods already described; see Chap. v. In other cases the method is useful if we wish to ascertain the forces in a few selected members or to check results obtained graphically.

96. Method of Sections. If in the framed cantilever shown in Fig. 155 we imagine the cantilever to be severed at the section X–X and the outer portion removed, we can consider the part played by the forces acting in the three severed members AB, AE and FE, in supporting the forces acting on the cantilever beyond the section. A little thought should make it clear that the two loads W_1 and W_2, which acted upon the cantilever to the right of X–X, must have been balanced by the three forces acting in the three severed members. In other words this system of five forces must have been a system in equilibrium to which the usual conditions of equilibrium would apply, so that we have:

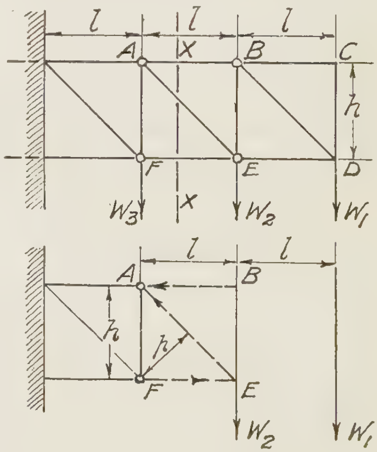


Fig. 155. Method of sections.

$$\Sigma V = 0,$$
$$\Sigma H = 0,$$
and
$$\Sigma M = 0 \quad \text{(see para. 10).}$$

It is possible to use these expressions and to write down three equations from which, if no more than three members have been severed, we can ascertain the unknown forces acting in them; but in practice, instead of proceeding in this way it is usually easier to write down three equations as equations of moments about some convenient point or points. In triangulated structures the point common to two of the forces (which intersect at a joint) may be taken in two of the cases. Any other point may be taken in the third case, but a joint lying in the line of one of the forces is usually selected as most convenient.

Taking the case shown in Fig. 155 the procedure would be as follows (see second figure, in which the outer portion beyond the section X–X has been removed).

(*a*) Since the force in AB and that in AE pass through point A, and therefore have no moment about it, if we take moments about A we have

$$\text{Force in } FE \times h = (W_2 \times l) + (W_1 \times 2l), \qquad \ldots\ldots\text{(a)}$$

from which we can find the magnitude of the force acting in FE.

(*b*) Similarly, taking moments about point E, through which the force W_2 and both AE and FE pass, we have

$$\text{Force in } AB \times h = (W_1 \times l), \qquad \ldots\ldots\text{(b)}$$

from which we can find the force in AB.

(*c*) Finally, taking moments about F, through which only FE passes, we have, where p is the perpendicular distance from F to AE,

$$(\text{Force in } AE \times p) + (\text{force in } AB \times h) = (W_2 \times l) + (W_1 \times 2l), \qquad \ldots\ldots\text{(c)}$$

from which, knowing the force in AB (see (b)), we can find the force acting in AE.

(**Note.** If in the expression (c) we should obtain a negative result, it would indicate that the force acting in AE acted in the opposite direction to that which we had assumed in writing down the equation.)

(*d*) At any joint, such as A in Fig. 155, the total moment of the external forces about that joint is equal to the bending moment at the joint. Thus we could write the equation (a) as

$$\text{Force in } FE \times h = B_a.$$

In using this latter method it is important to obtain the bending moment at the correct section. Thus to find the force in bar AB, the bending moment at section BE must be taken, i.e. *the section at which no inclined member is attached to* AB *in the bay being considered. Then*

$$\text{Force in } AB \times h = B_e. \qquad \ldots\ldots\text{(d)}$$

Experiment. The correctness of the above analysis may be verified by means of the apparatus described in para. 79; see also Fig. 156 (B). The outer and inner bays are made rigid, and the forces acting in the members AB, AE and FE are obtained in the manner already described, by replacing the bars successively by others capable of measuring the forces acting in them; see para. 79. For example, Fig. 156 (B) shows the

arrangement for finding the force acting in bar AE, the length of AE being adjusted until points A, B and C lie in the same horizontal line.

The results so obtained should then be compared with those obtained by calculation or graphically. The inner bay may be similarly dealt with though no new point arises which makes it necessary. The load W_3 does not affect the forces in the centre bay.

97. Forces in a framed girder obtained from the shear force diagram. If in Fig. 156 (B) we consider what is the *vertical force* acting over the second bay from the end we see that it must be equal to the sum of W_1 and W_2, but this is equal to the shear force S_b (see Fig. 156 (A)) over that bay. Now, since the members AB and EF are horizontal, the forces in them have no vertical component. Hence, since AE is the only inclined member in the bay, *the vertical component of the force in* AE *must be equal to the shear force acting over that bay.* This relation is true for all framed structures whose top and bottom members are parallel and, as we shall presently show in a series of examples, it may be used as the basis of a method of computing the forces acting in all the members of such structures.

(**Note.** The method can be extended to include framed structures of any outline, but such applications are considered to be beyond the scope of this volume.)

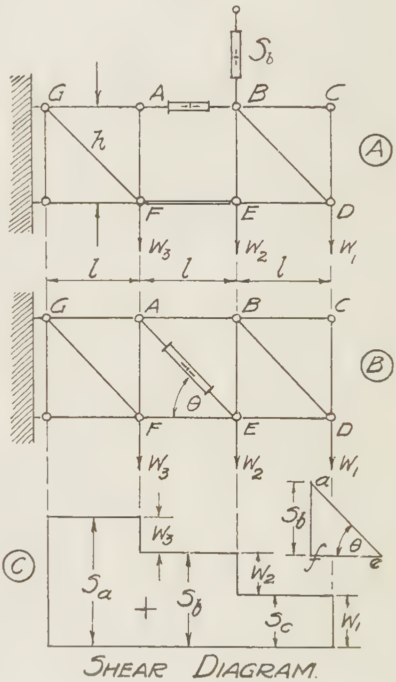


Fig. 156.

Experiment. In the preceding experiment we found the magnitude of S_b, which is the shear force acting over the centre bay, see Fig. 156 (C), and this is clearly equal to $(W_1 + W_2)$.

We also obtained the magnitude of the force acting in the inclined member AE. If the above relation holds, then the vertical component of the force in AE must be equal to S_b. In Fig. 156 (C) the small triangle aef has its sides parallel to the lines AE, EF and FA in the space diagram of the cantilever. If the angle aef be θ, then we have that

$$af = ae \times \sin\theta, \quad \text{or} \quad ae = \frac{af}{\sin\theta}.$$

But *aef* may be used as a force triangle for the force in AE, together with its vertical and horizontal components, hence

$$\text{Force in } AE = \frac{\text{shear force across the centre bay}}{\sin\theta}.$$

Or,

$$\text{Force in diagonal} = \frac{\text{shear force across bay}}{\sin\text{ (angle of inclination)}}. \qquad \text{......(i)}$$

If, as in this experiment, the angle θ is 45°, then, since sin 45° is $1/\sqrt{2}$, we have

$$\text{Force in } AE = \text{shear across bay} \times \sqrt{2}. \qquad \text{......(ii)}$$

(It is of interest to note that the force in bar EF will be greater than that in AB, by an amount equal to the horizontal component of the force in the inclined member AE.)

Forces acting in the verticals of a framed girder. If in the experiment described above the forces acting in the verticals are also obtained, it will be found that *the force acting in each vertical member is equal to the shear force acting over the adjoining bay.*

Some difficulty may, however, be experienced in deciding whether the shear value in the bay to the right or to the left of the member should be read. The difficulty is resolved by noting how the sloping member is attached to the vertical member. For example, take the vertical BE in Fig. 156, to which the sloping member BD is attached at the top. At the joint B the vertical member BE must supply the whole of the vertical force to balance the vertical component of the force in BD, since the other two members AB and BC are horizontal. But the vertical component in BD equals the shear force S_c over the first bay, therefore the force in the vertical equals S_c. Similarly the force in the vertical AF will equal the shear force S_b acting over the second bay.

98. Footbridge truss or girder with parallel top and bottom members. The truss, which is shown in Fig. 157, carries equal loads at each of the bottom joints, these joints dividing the length into four equal portions. The height of the truss is equal to the length of one bay, the diagonals being therefore inclined at 45°. (**Note.** The actual dimensions of the truss are not given, being unnecessary for our purpose.) The shear force diagram, which should not need explanation, is given below the space diagram for reference.

Vertical forces and components. Starting at the right-hand end we may write *vertically* across the diagonal FE the magnitude of the shear force in the first bay, viz. 3 tons; *this is the vertical component of the force acting in* FE. Since the diagonal is in compression—if FE were cut then the points F and E would move towards each other—we prefix the "3 tons" with a plus sign. (A minus sign is used in the case of tension.) In all such trusses the diagonals which *slope up towards the centre are in compression* under symmetrical or uniform loading, and *in tension when sloping downwards towards the centre.*

The same method is followed in the second bay, where we write "+ 1" vertically across the diagonal GD.

The force in the vertical FD must be $-$ 3 tons, since it has to balance the vertical component of the force in the diagonal FE. Likewise it must balance the vertical component force of 1 ton in the diagonal GD together with the load of 2 tons at D; this again gives $-$ 3 tons and is a check upon the first estimate.

The force in the vertical CG must clearly be $-$ 2 tons, since the horizontal members BC and CD can take no part in supporting the load of 2 tons at C.

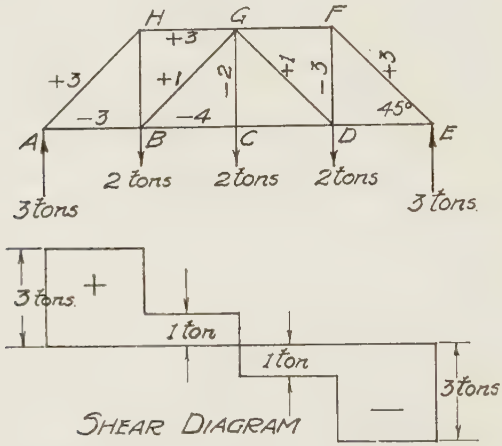


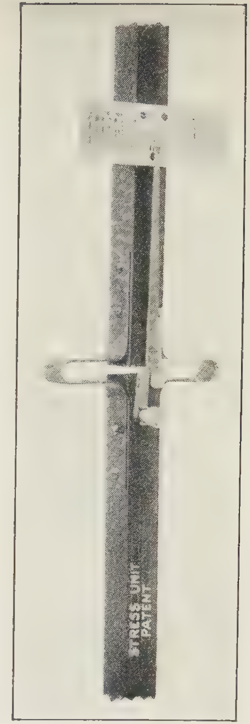
Fig. 157. Forces acting in a footbridge truss.

Horizontal components. Since the truss and the loading are both symmetrical, we need not trouble to write any more vertical components on the diagram and, proceeding this time from the left-hand end, we may now write down the horizontal components of the forces in the diagonals.

Since the diagonals are all inclined at 45°, the horizontal components of the forces acting in them must be equal to the vertical components of the forces acting in corresponding members. Thus it follows that the horizontal component in the diagonal AH is + 3 tons, while in the diagonal BG it is + 1 ton; these magnitudes are written *horizontally* across the member.

Forces in horizontal members. These may now be readily obtained from a consideration of the horizontal components of the forces in the diagonals. Thus the force acting in HG must be + 3 tons since it has to resist the horizontal component of the force in AH. The force in AB is likewise one of 3 tons but it is a tensile force and hence we write $-$ 3 tons against AB. Since at joint B the force in AB pulls to the left while the horizontal component of the force in BG pushes to the left, then the force in BC must be equal to the sum of these two forces, hence we write $-$ 4 tons against BC.

Fig. 157 A. Forces in framed structures—an experimental **N** girder.



Detail of stress unit.

Note. The experimental girder illustrated above has been specially devised to enable the forces acting in a number of members to be measured at the same time. Under load the leaf springs deflect slightly, the magnitude and nature of the load being indicated on the scale, which has been calibrated under test. The pointers can be adjusted to zero before any external loads are applied.

Forces in the diagonals. These may now be obtained by the use of the expression (ii) in para. 97, from which we have

$$\text{Force in } EF = 3 \text{ tons} \times \sqrt{2}$$
$$= 4{\cdot}24 \text{ tons.}$$

This result may also be obtained by calculation from the horizontal and vertical components; see the next Example. The latter method is useful where the inclination of the diagonals is fixed by the dimensions and not given in degrees.

Experiment. Such a truss as that shown in Fig. 157 may be fitted up from the apparatus described in para. 79, or from that shown in Fig. 157 A. All the forces acting in at least one bay should be obtained experimentally and compared with the calculated values. After the next example has been worked the experiment may be repeated with unsymmetrical loading; see Fig. 134 A.

99. Example. *A special roof truss based on the Warren girder.* The type of truss indicated in Fig. 158 is occasionally used to support, over a wide span, a "north-light" or "saw-tooth" roof, the roof surface following the lines of the diagonals. The shorter pitches are glazed while the longer pitches are roofed in the usual way.

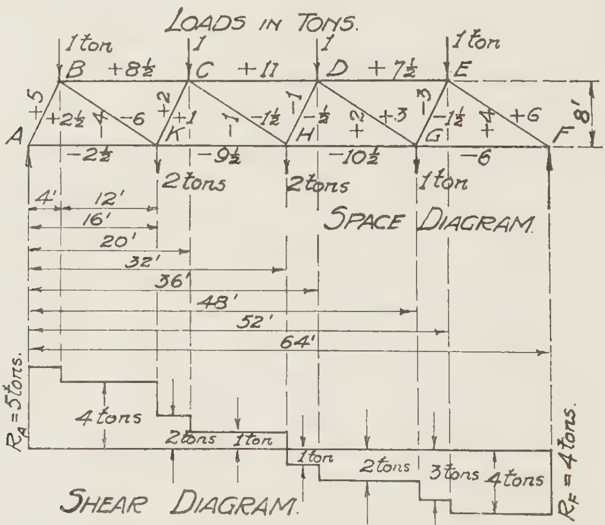


Fig. 158. Forces acting in truss supporting north-light roof.

For the sake of simplicity intermediate members and load points have been omitted. The loading is given in tons and, in order to make the case of more general interest, the loading is made irregular by introducing ceiling loads. The inclinations of the pitches are fixed by the dimensions of the truss.

Reactions. These should present no difficulty, though care is needed owing to the large number of load points. Taking moments about F, we have

$$64R_a = 1\times 60 + 2\times 48 + 1\times 44 + 2\times 32 + 1\times 28 + 1\times 16 + 1\times 12,$$

from which $$R_a = \frac{320}{64} = 5 \text{ tons.}$$

Similarly

$$64R_f = 1\times 52 + 1\times 48 + 1\times 36 + 2\times 32 + 1\times 20 + 2\times 16 + 1\times 4,$$

from which $$R_f = 4 \text{ tons.}$$

From these values the shear diagram can be readily constructed and should need no additional explanation; see Fig. 158.

Vertical components. These may be written direct from the shear force diagram, commencing with the end diagonals. The work is simplified by the absence of vertical members. The two inclined members intersecting at H clearly support between them the load attached at that point, they must therefore be in tension.

Horizontal components. The ratio between the vertical and the horizontal components of the forces in the inclined members is the same as the ratio between their vertical and horizontal dimensions; thus, the horizontal component of the force in AB is $(5 \times 4/8)$ tons or $+ 2\frac{1}{2}$ tons. Similarly the horizontal component of the force in BK is $-(4 \times 12/8)$ tons, that is -6 tons. These and the horizontal components of the other diagonals—obtained in a similar manner—are written horizontally across the diagonals.

Forces in horizontal members. At joint A, to produce equilibrium, it is clear that the member AK must supply a force of $2\frac{1}{2}$ tons acting away from the joint, to balance the horizontal component of $2\frac{1}{2}$ tons acting towards the joint. The force in AK is therefore $-2\frac{1}{2}$ tons.

At joint B the horizontal component of the force in AB is pushing towards the right, while the horizontal component of the force in BK is pulling towards the right. The force in BC must therefore be a compressive force equal to the sum of these two, that is $+8\frac{1}{2}$ tons.

At the joint K there are two horizontal components pulling towards the left—in AK and BK—while in KC there is a third horizontal force pushing towards the left. The force in KH must therefore be a tensile force equal to the sum of these three, that is $-9\frac{1}{2}$ tons.

The remaining forces may be obtained in a similar manner. As a check a few of the forces may be calculated independently by starting from the right-hand end.

Forces in the inclined members. These may be obtained from the "rectangular components" which we already know; see also para. 3.

Thus $$\text{Force in } BK = \sqrt{4^2 + 6^2} = \sqrt{52}$$
$$= 7{\cdot}21 \text{ tons.}$$

100. Example. *A French truss.* See Fig. 159. In addition to illustrating the application of the method of sections to a truss of different outline from those which we have already dealt with, the following example will show how the method may be used to supplement the graphical method in those cases where the latter method is difficult of application. The type of truss shown in

Fig. 159 (A) is a case in point. When the force diagram is being drawn it will be found that the work cannot proceed beyond the joints at C and M, since there are more than two unknown forces at each of these joints. The work can, however, proceed if one of the unknown forces be found; this can be readily done either (a) by the Method of Sections, or (b) the Method of Substitution.

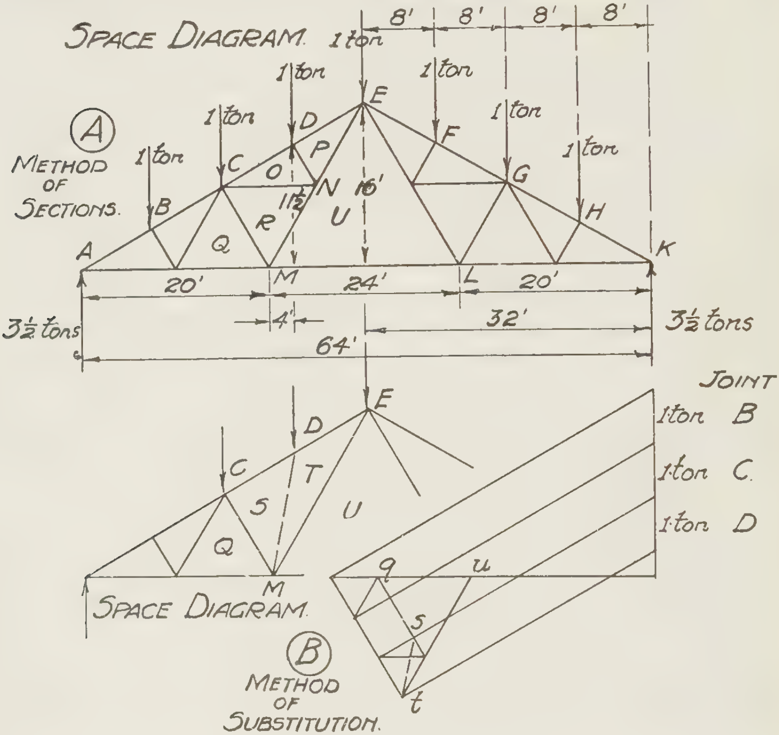


Fig. 159. Forces acting in a French truss.

(a) Solution by the Method of Sections. To find the force in the tie ML. If we imagine the tie ML to be severed by a vertical section passing through E, then, taking moments about E, the apex of the truss, we have

$$\text{Force in } ML \times \text{height of truss} = B_e;$$

see para. 96 (d).

But $\qquad B_e = (3\tfrac{1}{2} \times 32) - (1 \times 8 + 1 \times 16 + 1 \times 24)$
$\qquad\quad = 64$ ton ft.

Therefore $\qquad$ Force in $ML = 64/16 = 4$ tons.

This is sufficient to allow us to proceed with the graphical solution. If desired, however, the forces acting in the members DE and EN may be found in a similar manner, by assuming a vertical section to cut through the members to the left of E.

To find the force in DE. Taking moments about M we have, since EN passes through M,

$$\text{Force in } DE \times (\text{perp. distance from } M \text{ to } DE) = B_m.$$

Now $B_m = (3\frac{1}{2} \times 44) - (1 \times 4 + 1 \times 12 + 1 \times 20 + 1 \times 28 + 1 \times 36)$
$= 54$ ton ft.

Perpendicular distance of DE from M is 9 ft.

Therefore $\qquad$ Force in $DE = 54/9 = 6$ tons.

To find the force in EN. Take moments about D, then we have

$$B_d = (3\tfrac{1}{2} \times 40) - (1 \times 8 + 1 \times 16 + 1 \times 24 + 1 \times 32)$$
$$= 60 \text{ ton ft.}$$

The perpendicular distance of ML from D is $11\frac{1}{2}$ ft., while the perpendicular distance of EN from D is 4 ft. Both the members ML and EN are in tension (this is obvious from an examination of the truss); hence, noting the direction of the forces in these two members about the point D, we have, since force in ML is 4 tons,

$$(4 \times 11\tfrac{1}{2}) + (\text{force in } EN \times 4) = B_d = 60$$

or $\qquad (\text{force in } EN \times 4) = 60 - 46;$

that is, $\qquad$ Force in $EN = 14/4 = 3\frac{1}{2}$ tons.

The above method, which is of quite general application, can be applied to any case for which the dimensions of the structure are fully given or can be found. The following method is only applicable where the forces in the members of this particular type of truss are being found by graphical methods

(*b*) **Solution by the Method of Substitution.** The method is also known as "Barr's method". In it we substitute a new member DM for the two members DN and NC, see Fig. 159 (B), and thus obtain in this portion of the truss a series of joints at each of which there are not more than two unknowns, so that the graphical solution may proceed.

Having drawn the new member DM, see lower space diagram, proceed up to point q on the force diagram and then draw qs through q parallel to Q–S (using new space lettering in diagram). Through point s thus obtained, draw st parallel to S–T and thus obtain point t. Through t draw tu parallel to T–U and thus obtain u on the horizontal through q. Since u is a point on the completed force diagram, we can now replace member DM by the original members CN and ND and complete the force diagram by working backwards from u.

Problems XI

(**Note.** The results obtained in the following problems may be checked by experiment in the manner described in paras. 96 to 98.)

1. If in Fig. 155 the weight $W_1 = W_2 = 14$ lbs. and the distance $AB = BE = BC = 10$ ins., calculate the magnitude of the forces acting in the members AB, AE and FE.

2. Using the same values as are given in Prob. 1, find the forces acting in the vertical members BE and AF and check the force acting in the inclined member AE. (Use the methods given in para. 97.)

3. The truss shown in Fig. *A* is known as an N- or Pratt truss. It is occasionally used to support a series of roof trusses in the manner shown in the small sketch, where a large area is to be roofed over without intermediate supports. Assuming for simplicity that the truss is loaded as shown in Fig. *A*, find the magnitude of the forces acting in the members *AF*, *EB*, *EF*, *FB* and *DE*. (See experimental truss in Fig. 157 A.)

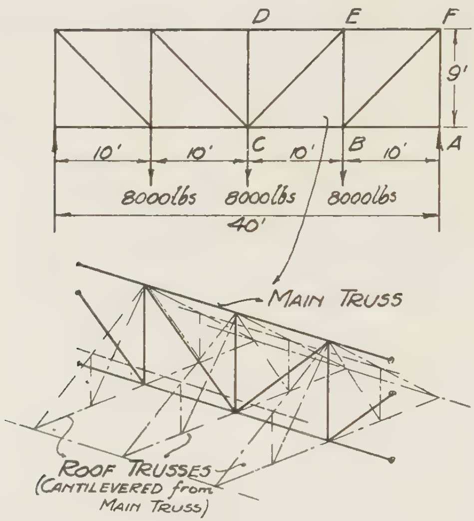


Fig. *A*. N-truss supporting roof.

4. If the truss shown in Fig. 157 has two additional loads of 2 tons each, placed at points *C* and *D* respectively, find the forces acting in the members radiating from point *G*.

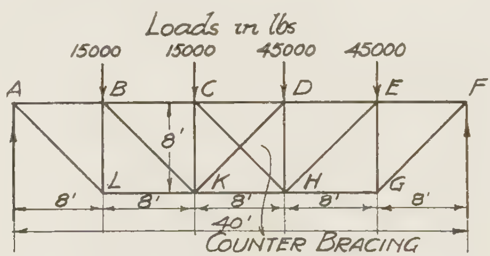


Fig. *B*. Truss supporting gallery.

5. The truss shown in Fig. *B* is used to support a gallery or balcony resting on its upper surface. As the gallery may be only partially loaded, the centre panel is counterbraced to deal with the unsymmetrical loading so produced. The inclined members are only capable of resisting tensile stresses. For the loading indicated find the forces acting in the members forming the centre panel.

6. The truss shown in Fig. C is known as a Howe truss and is suitable for construction in timber and steel. The truss is to be used to carry a flat roof above and a ceiling below. For the loading given, calculate the forces acting in the members AB, BC, AC and CD.

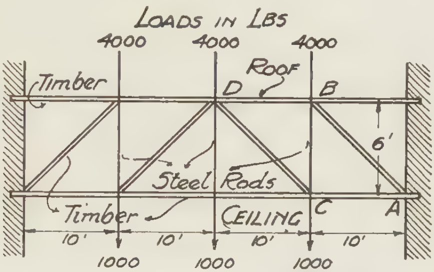


Fig. C. Timber and steel Howe truss supporting flat roof and ceiling

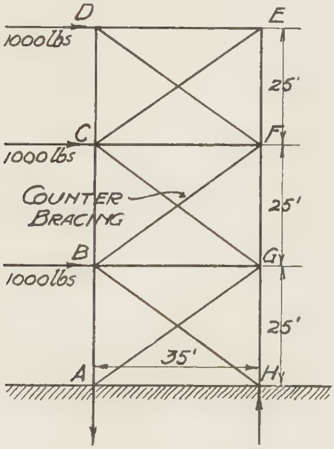


Fig. D. Side of derrick tower subjected to wind pressure.

7. Fig. D gives the dimensions of one side of a derrick tower, which is divided into three equal bays by horizontal members, each bay being counterbraced with stays which are only capable of resisting tensile forces. If the effect of wind pressure on the tower is to subject this frame to horizontal forces at the joints B, C and D as shown, calculate the forces acting in each horizontal member (excluding AH), and in each inclined member brought into action.

CHAPTER XII

STRESS AND ELASTIC STRAIN. STRENGTH AND ELASTICITY OF STRUCTURAL MATERIALS

101. The deformation of structural materials. It is a matter of common experience that forces may be transmitted through solid materials and that those materials are *deformed* or *strained* in the process, being either lengthened, shortened, twisted or bent. We also know that, if the forces so transmitted be not excessive, then the deformation which is produced will disappear on the removal of the forces. To this latter property has been given the name of **Elasticity** (see Vol. I, Chap. VI).

In certain building materials, such as steel or stone, the amount of deformation due to the application of external forces may be very slight, so slight in fact as to be invisible to the unaided human eye; but, by the use of powerful testing machines, together with highly sensitive measuring instruments, we know that even the most rigid materials are strained in this way. By the same means we are able to measure the amount of that strain (even in cases where it is extremely small), to ascertain the magnitude of the force producing the strain, and to prove that in common with more flexible materials they also possess this property of elasticity. This and similar facts have furnished the experimental basis upon which has been built up scientific knowledge concerning the relations between stress and strain in "elastic" materials. These relations form an important and essential link between the work which we have already done on the distribution of forces in the members of structures and that which we propose to do, in the later stages of this volume, on the strength of structural materials and the manner in which applied forces are resisted by them.

Though the subject of elasticity must for practical purposes be closely associated with experimental work, it is a subject which has been developed during the last 250 years with great fullness on the mathematical side in relation to an imaginary and ideal elastic substance. While it is not necessary in a sectional treatise of this kind to enter into the more complex aspects of the subject, it is important that we should be acquainted with some of the elementary conceptions of it, before proceeding to deal with practical building problems in which a consideration of elasticity is involved.

102. Stress. When a body is held in equilibrium by a simple system of external forces acting upon it, then it is clear that the effects of those forces must be transmitted through the material of the body. If we consider a plane surface within the body, at any convenient section of it, then equal but opposite forces must be produced within the material of the body and must act on either side of such a section, for, if not, then the body would either be torn apart or crushed by the relative movement which would take place, and equilibrium would be destroyed.

These equal and opposite forces can only be supplied by the material of the body, which is therefore said to be in a "state of stress". Hence we may say that *Stress is the resistance set up within the substance of a body which is transmitting external forces.*

Though stresses may, as we shall see, be very complex, they can be analysed into three simple stresses, viz. tension, compression and shear stresses.

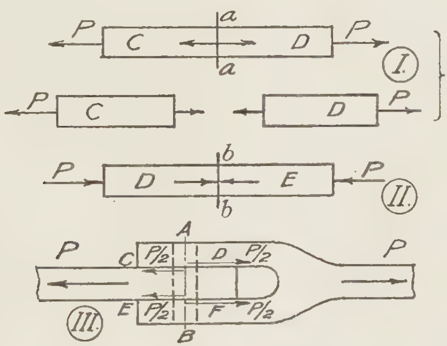


Fig. 160.

Tensile stress. If CD in Fig. 160 (I) be part of a tie bar which is subjected to a pull P at either end, then the stress at the section a–a, taken at right angles to $P\ P$, is a tensile stress which is resisting the tearing of the portion C from the portion D. The resistance of the material across the section a–a must evidently be such that the portion C is held in equilibrium by the force P to the left and an equal and opposite force acting at the other end; this latter force is the result of the stressing of the substance of the bar. The equilibrium of the portion D may be similarly stated.

When, as in this case, the stress is a simple one, the *intensity of stress* or the *stress per unit area* is obtained by dividing the total force acting across the section by the area of the section; thus, if the area of the section at a–a be A, we have:

$$\text{Intensity of stress} = \text{stress per unit area} = \frac{\text{total force}}{\text{area}},$$

or
$$\text{Stress} = \frac{P}{A}. \qquad \text{......(i)}$$

If P be measured in pounds and A in square inches, then the stress will be given in pounds per square inch or, abbreviated, "lbs. per sq. in.".

In future we shall use the word "stress" to mean "intensity of stress" or "stress per unit area", the force acting across the whole area being referred to as the "total stress" or merely "the force".

In those cases where the stress is not uniformly distributed over an area the "stress at a point" may be taken to mean the stress over a small area situated at that point.

Compressive stress. If in the case shown in Fig. 160 (II) the bar DE is subjected to the action of two equal forces P acting at either end and towards each other, then the stress at the section b–b will be a compressive stress and its intensity will be given as before by

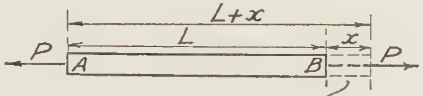
$$\text{Compressive stress} = \frac{P}{A}.$$

Shear stress. A case of shear stress is illustrated in Fig. 160 (III). A forked tie bar is connected to another by a pinned joint at AB. A pull P is applied to each of the portions of the bar. If, as we may assume, half of the pull P is transmitted to the pin by each half of the fork, then at the sections CD and EF the pin AB is subjected to a force of $P/2$ which tends to slide one portion of the pin past the other; such a stress is known as a shear stress and its intensity is obtained by dividing the total force by the area of the section across which it is acting, i.e. in this case the area of the cross section of the pin. If this be A, then

$$\text{Shear stress in pin, at } CD \text{ and } EF, = \frac{\text{force}}{\text{area}} = \frac{\frac{P}{2}}{A}.$$

103. Strain. All bodies alter in shape when subjected to stress; *this alteration in shape is known as "Strain"*. For every kind of stress there is a corresponding strain. Thus the tie bar CD shown in Fig. 160 will be lengthened under the action of the tensile stress, while the bar DE will be shortened under the compressive stress. (Shear stress and strain is dealt with in Chap. xv and will not be further defined here.)

The term **strain** may with advantage be used to indicate the *extension (or compression) per unit of length.* Thus in Fig. 161 if



x = Total Strain or Extension.

Fig. 161.

the original length of the bar AB is L units and, after being subjected to a pull P, it extends to $(L + x)$ units in length, then the total extension is evidently x. If then we define "strain" as "extension per unit length", then

$$\text{Strain} = \frac{x}{L} = \frac{\text{total extension}}{\text{original length}}. \qquad \text{......(ii)}$$

104. Limits of elasticity. It may be shown experimentally for most materials that if the stress due to applied forces does not exceed a certain magnitude, then the strains which are produced will disappear entirely on the removal of the stress. The limiting stress in such a case is known as the **Elastic Limit.** At any stress below the elastic limit stress, such a material is said to be truly elastic.

If the material be subjected to a stress above the elastic limit stress, all the strain does not disappear; the strain remaining after the removal of the stress is known as the **Permanent Set.**

105. Hooke's Law. One of the most fundamental laws of elasticity was that first enunciated in the seventeenth century by Hooke, who showed that for many materials *the extension was proportional to the force producing it.*

For our present purpose we can now put this in a more general form as follows: **Within the elastic limits for any material the strain is proportional to the stress.**

106. Coefficients of elasticity. Young's Modulus. If Hooke's Law be true for a material, and the strain is proportional to the stress, then we may express the relation in the following form:

$$\text{Stress} = \text{Strain} \times \text{Constant}.$$

The constant in the above expression is known as the **Coefficient or Modulus of Elasticity**; hence we have that

$$\text{Coefficient of elasticity} = \frac{\text{stress}}{\text{strain}}. \qquad \text{......(iii)}$$

The coefficient of elasticity varies for the different kinds of stress and strain and also for the various materials. Fortunately for our purpose *this coefficient is practically the same in tension and compression for most structural materials.* This is the coefficient with which we shall be most concerned in this volume; it is known as Young's Modulus, after the English physicist who first established it. It is usually expressed in abbreviated form by the letter $\boldsymbol{E}$.

The value of Young's Modulus for any material is ascertained experimentally, using the equation,

$$\text{Young's Modulus} = \boldsymbol{E} = \frac{\text{stress}}{\text{strain}}. \qquad \text{......(iv)}$$

It should be noted that, since "strain" expresses deformation as "a fraction of unit length", while stress is measured as "a force per unit area" (e.g. lbs. per sq. in.), then the "coefficient" must be expressed in the same terms as the stress, i.e. force per unit area.

(The Shear or Rigidity Modulus (G) is referred to in Chap. XV. The Bulk Modulus (K), which is not of importance to us at this stage, applies to the case wherein a body is subjected to three mutually perpendicular and equal stresses, being reduced or enlarged in volume thereby.)

107. Experiment. *To find Young's Modulus for a piece of iron wire.* In Vol. I, Chap. VI, particulars of an experiment are given in which gradually increasing loads were suspended from a length of iron wire.

Fig. 162. Relation between load and extension in an iron wire.

The increases in length over an original length of 84 ins. were carefully measured and a graph plotted showing the relation between the load and the extension. For convenience this graph is reproduced in Fig. 162.

Since the graph is a straight line within the limits chosen, the strains are evidently elastic strains, since they bear a constant relation to the stresses by which they are produced.

The diameter (d) of the wire was 0·05 in.

Reading off the force at any convenient point, say 40 lbs., we find that the resulting extension was 0·059 in. Hence we have

$$\text{Stress at 40 lbs.} = \frac{\text{force}}{\text{area}} = \frac{40}{\frac{\pi d^2}{4}} = \frac{40 \times 4}{\pi \times 0\cdot 0025} \text{ lbs. per sq. in. } \ldots\ldots\text{(a)}$$

Also $$\text{Strain at 40 lbs.} = \frac{\text{increase in length}}{\text{original length}}$$
$$= \frac{0\cdot 059}{84}. \qquad \ldots\ldots\text{(b)}$$

Then

$$\text{Young's Modulus } (E) = \frac{\text{stress}}{\text{strain}},$$

which, from (a) and (b),

$$= \frac{40 \times 4}{\pi \times 0\cdot 0025} \div \frac{0\cdot 059}{84}$$
$$= 28{,}800{,}000 \text{ lbs. per sq. inch.}$$

108. Lateral strain. Poisson's Ratio. If a piece of rubber be subjected to a tensile stress it will be stretched or drawn out, and close observation will show that its lateral dimensions, i.e. its dimensions at right angles to the direction of the pull, are reduced. Correspondingly if a piece of rubber be compressed its lateral dimensions will be increased. In other words the strain which takes place in the direction of the principal stress is accompanied by certain lateral strains, usually opposite in nature, i.e. tensile in the first direction results in compression (or "negative extension") in the second and *vice versa*. Such accompanying strains are known to exist in most structural materials, though they are naturally very minute and difficult to measure directly. For each elastic material the amount of the lateral strain bears a definite ratio to the longitudinal strain. This ratio (usually between one-quarter and one-third) is called Poisson's Ratio,

or $$\text{Poisson's Ratio} = \frac{\text{lateral strain}}{\text{longitudinal strain}}.$$

It is of importance to note that Young's Modulus, since it is deduced from experimental results, takes this lateral straining into account. It is, in fact, a practical constant and is not based upon what might be termed "pure strain", i.e. strain in one direction unaccompanied by strain in another direction.

109. Ultimate Strength. Factor of safety. If a piece of material be stressed in a testing machine until it fails, the greatest stress

which it carries is known as the **Ultimate or Breaking Strength.** As will be seen when studying the testing of materials, it is not always easy to decide the value of the ultimate strength, and certain conventions have grown up which must be taken into account. For our present purpose, however, we may assume tha for each material it has a definite value.

Working Stresses. It is the common practice when designing structures, to use for each material certain definite stresses which long practice has shown to be safe stresses for those materials. These are known as **Working Stresses.**

Factor of safety. The ratio between the ultimate strength and the working stress is known as the Factor of Safety or

$$\text{Factor of safety} = \frac{\text{ultimate strength}}{\text{working stress}}. \qquad \ldots\ldots(\text{v})$$

The term "factor of safety" is also used at times to express the ratio between the theoretical (or actual) load at which a structure might be expected to (or does) fail, and the load which it is designed to carry. A common value given to the factor of safety is 4, but so many considerations affect this value, most of them of too complex a nature to be discussed in this volume, that it would be unwise in the majority of cases to use this method alone for deciding the magnitudes of the working stresses.

110. A complete tensile test of a piece of mild steel. Of all the modern structural materials steel is the one which approaches most nearly to the ideal elastic material. Practically all its characteristics are well-defined. It has a well-marked elastic limit, below which it is almost perfectly elastic. Beyond the elastic limit its nature changes steadily and it becomes more and more *plastic* —strain increasing more rapidly than stress —up to the point of failure. These facts are well brought out by the stress-strain curve, shown in Fig. 163, which is obtained when a specimen of mild steel is tested in tension in the manner described in para. 113.

On the graph shown in Fig. 163 the portion from A to B represents that part of the test during which the relation between stress and strain is constant, AB being a straight line. The point B indicates the **Elastic Limit** for this specimen. Beyond B the line curves steadily for a little distance and then suddenly moves in an almost horizontal direction. The load or point at which this occurs is known as the **Yield Point** and the accompanying stress as the **Yield Stress.** The strain is relatively so large at this point that it is possible to detect the load at which it occurs by the large and sudden movement of the lever of the testing machine.

Being a clear and easily noted indication of the limit of ordinary elastic conditions, it is generally known as the **Commercial Elastic Limit.**

Owing to the large strains which take place from C onwards, it has been necessary to draw the complete graph to a very small scale. The portion ABC has therefore been re-drawn to a larger scale, see $AB'C'$, Fig. 163, so as to indicate more clearly the nature of the relations between stress and strain over this section. This graph should make quite clear the difference between the true elastic limit and the commercial elastic limit.

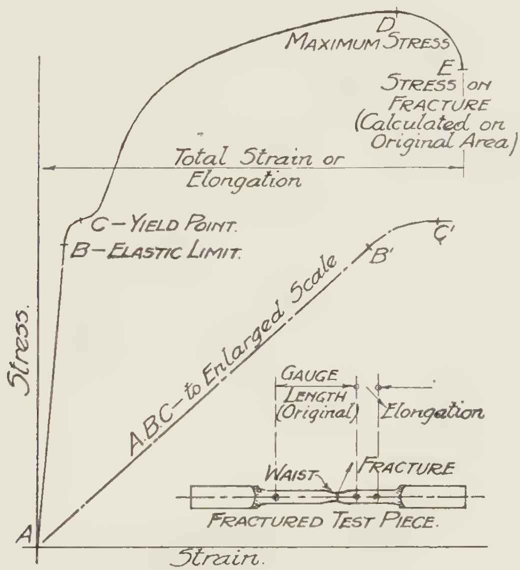


Fig. 163. Stress-strain graph for a tensile test on a piece of mild steel.

Beyond C the strains increase more rapidly than the stress, the graph being thus curved. This goes on until, at D, the material becomes almost perfectly plastic, strain increasing with only slight increases in the stress.

Up to D the diameter of the test piece will have been reduced owing to the lateral straining to which we have already referred. At or near D, however, there is a sudden and marked reduction in the diameter of the test piece over a short length, a "waist" being formed which is clearly visible to the eye. This is shown by the small sketch in Fig. 163, in which the two pieces of the test piece have been placed together again after fracture. This local reduction is so considerable that the actual load at E required to fracture the

test piece is less than the load carried at D. If, therefore, the stress at fracture be calculated by dividing the fracture load by the *original* cross-sectional area of the test piece, then the stress at fracture would appear to be less than that borne by the material at D. If, however, the load carried at fracture be divided by the actual area of the specimen at the reduced section, it will give a stress which is greater than that borne at D. Since, in practice, it would be difficult to measure the dimensions of the fractured surface of the test piece, the so-called *ultimate strength* is found by dividing the *maximum load* carried by the specimen by the *original area* of the cross section of the specimen,

or $$\text{Ultimate strength} = \frac{\text{maximum load}}{\text{original area}}. \qquad \text{......(vi)}$$

111. Elongation and reduction in area. The total strain which a tensile test piece shows in such a test as that described above is known as the **Elongation**. We have seen that the elongation includes the steady strain which took place in the early part of the test, as well as the sudden "local" straining which took place when the "waist" was formed. Since this latter strain is both considerable *and* localised, and is not seriously affected by the total length of the test piece, it is important, if comparable results are to be obtained between one specimen and another, that the length over which the elongation is to be measured should be specified. A "gauge length" of 8 ins. is commonly adopted or, alternately, this length is fixed in proportion to the diameter of the specimen (usually at least eight times). Whatever length be selected, it is marked on the specimen by means of a steel punch. The first and smaller strains are measured by means of an extensometer, as already noted. The greater strains which take place after the elastic limit has been passed, as well as the total elongation, may be measured directly with a rule.

If the elongation measured as described above be considerable, then the material is said to be **Ductile**. An elongation of 20 % is a common value for structural (mild) steel, a value which is not reached by any other structural material.

Since, as will be obvious, the one is related to the other, it will be seen that the reduction of the area of the cross section of the specimen at the point of rupture is also a measure of its ductility. A common value for this reduction of area for structural steel is approximately 60 %.

Materials such as cast iron, certain high carbon steels, and steel which has been "hardened" by heat treatment, show only a slight elongation and no measurable reduction in area and are therefore usually classed as **Brittle** materials. The same remarks apply to

materials such as brick, stone and concrete, though it is obviously impracticable to apply to them such tests as have been described above.

112. The elastic limit stress. A mild steel, such as that used in structural work, is characterised by a comparatively low elastic limit stress (usually about three-fifths of the ultimate strength). As we have already seen, the "elongation" in the case of such a material consists mainly of the strain which takes place at stresses beyond the elastic limit stress. We may therefore say that *a ductile material is characterised by a relatively low elastic limit stress and a considerable extension at higher stresses.* With most steels the elastic limit stress may be raised by "cold-working" or, what is probably the same thing, by "over-strain". If a test piece be subjected to a stress above the elastic limit stress, it will of course show a certain amount of permanent set. If, however, it be re-stressed, it will be found to exhibit the property of elasticity (that is, stress and strain will be proportional to each other) up to the highest stress to which it has already been subjected; this latter stress has in fact become the new elastic limit. (This process cannot of course be repeated indefinitely, as will be readily seen if we bend a bar of mild steel to and fro for a number of times; in the end it will break, though the amount of bending may not have been greatly increased.) This is a valuable property in connection with structural work. If, by reason of inaccurate construction or fixing, a member of a steel structure be subjected to a severe local strain which produces a stress in excess of the elastic limit stress, then, provided of course that the stress be not altogether too high, the material will adjust itself to new conditions and continue to function satisfactorily, so long as it is not then subjected to further stresses beyond the new elastic limit so produced.

There is, however, this accompanying disadvantage that, as a rule, a high elastic limit stress is associated with a reduction in ductility. This is found to be so when the elastic limit is raised by cold-working, by heat treatment ("hardening" and "tempering"), or by an increase of the carbon content of the steel, though certain alloy steels have now been produced of considerable ultimate strength, which show both a high elastic limit stress and considerable ductility.

113. Experiment. *A tensile test of mild steel.* In para. 124 a complete bending test is described, which was carried out upon a mild steel beam. In order to find the strength of the steel and the correct value to use for E, the modulus of elasticity, the following tensile test was carried out on a piece of steel cut from the beam.

Testing machine. Though, owing to the relatively large diameter of the test piece, see Fig. 164, this test was carried out on a testing machine of considerable capacity, the testing machine illustrated in Fig. 165 should be capable of completing such a test, provided the specimen is turned down to such dimensions that the rupture or breaking load is within the capacity of the machine.

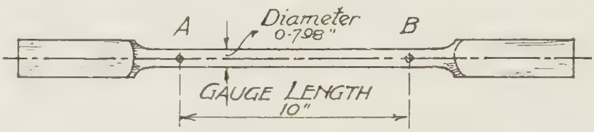


Fig. 164. Steel tensile test piece.

Test piece. This is shown in Fig. 164. The centre portion was turned down to a diameter of 0·798 in., thus giving a sectional area of ($\pi/4 \times 0{\cdot}798^2$) or 0·5 sq. in.

This portion was about 12 ins. in length in order to provide for a gauge length of 10 ins. (see note below on the Extensometer).

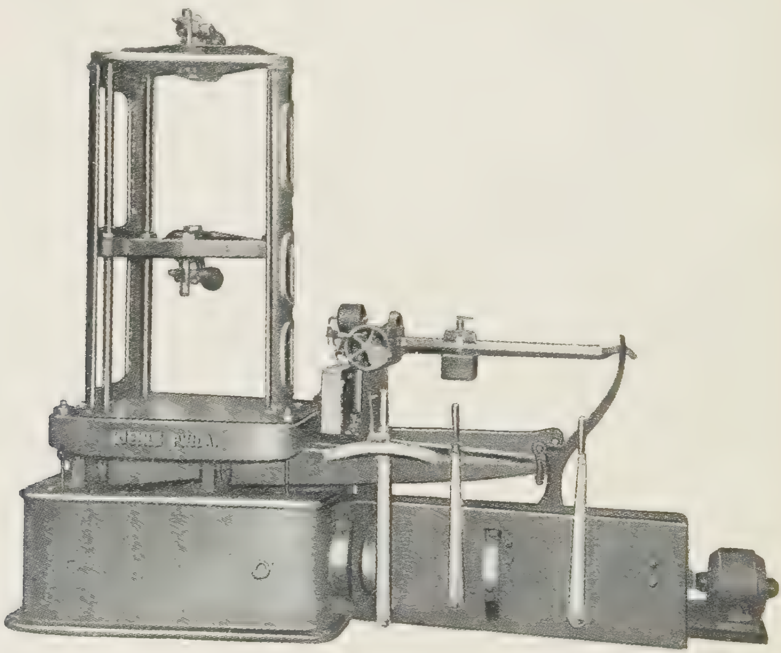


Fig. 165. Testing machine.

Ultimate strength. The test piece broke at a load of 12·1 tons, then

$$\text{Ultimate strength} = \frac{\text{rupture load}}{\text{original area}} = \frac{12{\cdot}1}{0{\cdot}5}$$
$$= 24{\cdot}2 \text{ tons per sq. in.}$$

Yield point stress. This occurred at a load of 7·8 tons.

Then $$\text{Yield point stress} = \frac{\text{load}}{\text{area}} = \frac{7{\cdot}8}{0{\cdot}5}$$
$$= 15{\cdot}6 \text{ tons per sq. in.}$$

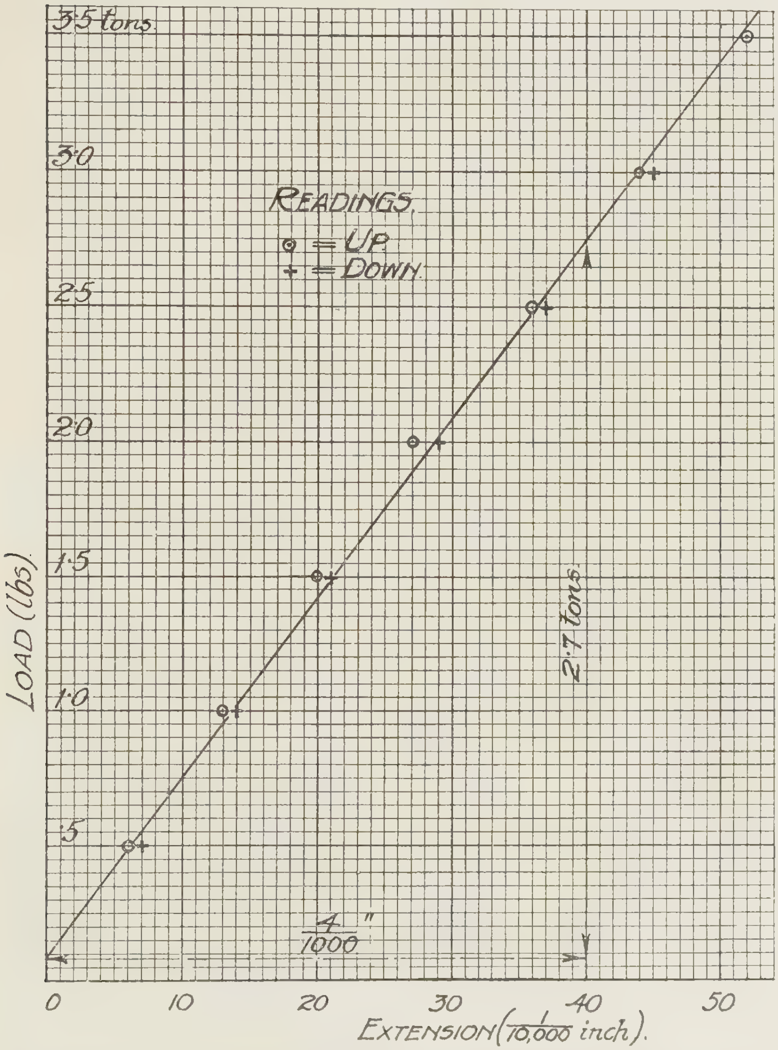


Fig. 166. Load-extension graph within the elastic limit.

To find *E*, using an Extensometer. An extensometer is an instrument used to measure small "extensions". Similar instruments are

used to measure compressive strains, and are referred to as "compressometers". The extensometer used in this experiment—and which can also be used to measure compressive strains—is illustrated in Fig. 175 (B) and described in para. 124. The advantages of this type of extensometer are that it is simple in construction and its action is easily understood. It can, however, only be used satisfactorily on large specimens, since a gauge length of 10 ins. is necessary; see Fig. 164. Changes in length of the test piece as small as 0·0001 in. can be read if the instrument is carefully set up and used.*

The results obtained in the present test are indicated in the table below. From these figures the graph shown in Fig. 166 has been plotted. In drawing this graph it will be found that the points obtained lie close to a straight line. Having drawn the straight line which appears to interpret the results as fairly as possible, this line is then used to furnish the values required to ascertain the magnitude of E.

Thus we find from Fig. 166 that, for an increase in extension of 0·004 in., there was an increase of load of 2·7 tons. Then

$$E = \text{modulus of elasticity} = \frac{\text{stress}}{\text{strain}}$$
$$= \frac{2{\cdot}7 \times 2240}{0{\cdot}5} \times \frac{1}{0{\cdot}004}$$
$$= 30{,}000{,}000 \text{ lbs. per sq. in. (approx.).}$$

Readings by extensometer (gauge length 10 ins.).

Load (tons) ...		0	0·5	1	1·5	2	2·5	3	3·5
Extensions in 1/10,000 in.	Up	0	6	13	20	27	36	44	52
	Down	1	7	14	21	29	37	45	52

Note. Readings were taken as the load was increasing ("Up"), and also as the load was decreasing ("Down"). The elastic limit stress was not exceeded.

114 Strength characteristics of structural materials other than steel. The question of the strength and elastic properties of the other structural materials need not be gone into in detail at this stage, since they will be dealt with more fully in Section III, in connection with the various types of construction. As will presently appear, the tests described in this chapter are not suitable for materials other than steel and wrought iron, and other tests are used, of which the "Compression Test", see Chap. XXIII, is perhaps the most important.

Materials such as cast iron, stone and concrete, all of which are weak in tension but strong in compression, have no well-defined elastic limit. Usually they show small amounts of permanent set at low stresses on being first loaded, practically no part of the

* Readers desiring more complete information on the subjects of testing machines and extensometers should consult Unwin's *Testing of Materials*, Popplewell and Carrington's *Materials of Construction*, or Smith's *Handbook of Testing Materials*.

stress-strain curve being really straight. It is thus usual to adopt some convention as to the portion to be used in determining the value of E; see stress-strain curve for concrete, Fig. 225.

The strength of steel and cast iron in compression (also of other materials) is dealt with in Chap. XXIII.

Timber, see Chap. XXV, is a material which is flexible but not ductile. It has, however, a fairly well-defined elastic limit. This is most readily obtained from a bending test, from which also the value of E may be most suitably obtained; see Exper. in para. 143.

Problems XII

1. If in Fig. 160 (III) the pull P is 90,000 lbs., calculate (a) the net area of the tie bar if the tensile stress is not to exceed 16,000 lbs. per sq. in., and (b) the diameter of the pin AB if the shear stress is not to exceed 12,000 lbs. per sq. in.

2. If the tie bar in Prob. 1 is 20 ft. long, find the total extension when the bar is subjected to the pull P of 90,000 lbs. if E is 30,000,000 lbs. per sq. in.

3. In the test illustrated by Fig. 163 the following values were obtained: Elastic limit stress 35,000 lbs. per sq. in., strain at point B' 0·0012 in.; breaking load 5·9 tons, original diameter 0·5 in., final diameter 0·34 in., gauge length 8 ins., final length between gauge points 9·5 ins. Using these values, find (a) the modulus of elasticity (E), (b) the ultimate strength, (c) the reduction in area, and (d) the elongation.

4. What would be the actual stress at fracture of the test piece described above in Prob. 3?

5. If a beam be made from the material tested in experiment, para. 113, and then loaded, the material will be strained differently at different layers. This strain was measured at two layers X–X and Y–Y; find the stress in each layer if the following results were obtained: (a) at layer X–X the original gauge length of 10 ins. extended to 10·0066 ins., (b) at layer Y–Y the original gauge length of 10 ins. shortened to 9·996 ins.

6. A short reinforced concrete column has on the section an area of concrete of 100 sq. ins. and an area of steel of 1·5 sq. ins. If on a length of 20 ins. the column is found to shorten by 0·006 in. when subjected to a certain load, calculate (a) the stress in the steel and the concrete, (b) the portions of the total load carried by the steel and by the concrete, and (c) the total load on the column. The value of E for the concrete may be taken to be 2,000,000 lbs. per sq. in., while that for the steel may be taken to be 30,000,000 lbs. per sq. in.

CHAPTER XIII

THE THEORY OF ELASTIC BENDING

115. Simple bending. In Chaps. VIII, IX and XII we considered the fundamental principles upon which the study of the present chapter is based.

Chap. VIII dealt with the equilibrium of forces in a loaded beam, and showed the general relations which exist between the external and internal forces.

Chap. IX explained methods of analysing the principal forces, and of estimating and representing shear forces and bending moments, special attention being given to the common cases of practical loading.

Chap. XII dealt with the property of elasticity and established the relations which exist between stress and strain in elastic materials.

The object of the present chapter is to study in detail the internal action of the material of a loaded beam, and to ascertain the distribution of the elastic strain and of the related stress in the fibres of the beam.

Experiment. In Chap. VIII we obtained a rough conception of the way in which the material of a bent beam is strained, by means of a short deep rubber beam (see also Fig. 167). Before bending, a horizontal line was marked along the middle of the depth of the beam, and also a series of equidistant vertical lines.

If a load be applied at the centre of the beam sufficient to produce a considerable deflection, see Fig. 167 (*b*), it will be seen, from the relative positions of the originally vertical lines, that the upper layers of the beam have been compressed while the lower layers have been extended.

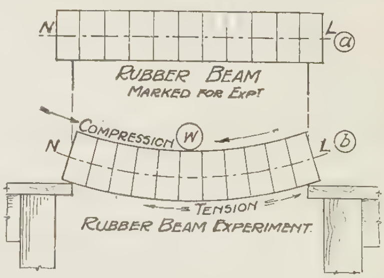


Fig. 167. Rubber beam.

Further, the amount of this change—or strain—will be found to be greater the farther the layer is from the middle layer NL.

Lastly, while the line NL has been bent in following the curvature of the beam, its actual length will be found to be unaltered. This layer would therefore appear to have suffered no longitudinal strain.

From this rough conception of what takes place in the material of a loaded beam we may now proceed to build up a more exact statement.

We shall limit our investigation in the first place to what is called "simple" or "pure bending". Such a case is shown in Fig. 168, in which a beam RS of uniform section is loaded with two equal loads, W and W, at points M and N, which are equidistant from the ends of the beam. If the distances RM and SN are both equal to l_1, then, since the reactions at each end are equal to W, the bending moment (B), between M and N, will be constant and equal to $W \times l_1$. Between these two points the shear force (S) will be zero. (The case is similar to the one dealt with in para. 88, the directions of the loads and reactions being reversed.)

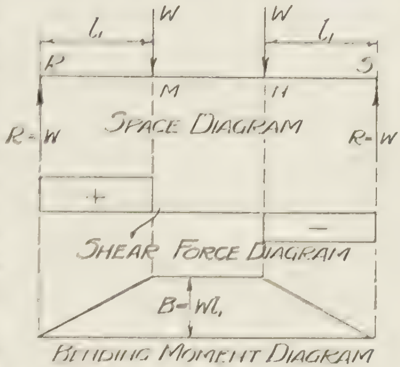


Fig. 168. Case of simple bending.

Any short portion of the beam between the two points M and N will be subjected only to two equal and opposite couples, each of which are equal to B, the bending moment, see Fig. 169. The effect of these applied couples will be to bend this portion of the beam to the form shown (exaggerated) in the second figure in Fig. 169.

116. *Assumptions in the theory of simple bending.* Before considering the straining actions which take place in a loaded beam, certain conditions and assumptions must be stated which, as the discussion proceeds, will be seen to be of fundamental importance.

Assumption I. That the beam has the same cross section throughout and is at least symmetrical about a vertical axis. (The majority of the beam sections are symmetrical both about their vertical and their horizontal axes but, for the present, we will limit our assumption to the case mentioned. The beam is assumed to bend in a plane parallel to that containing the vertical and longitudinal axes of the beam.)

Assumption II. That, for the material of the beam, strain is proportional to stress, i.e. the limits of elasticity are not exceeded; and the value of Young's Modulus (E) is the same in tension and compression.

Assumption III. That each layer in the beam is free to expand or contract laterally under stress, being unaffected by adjacent layers which may not be subjected to the same stress intensity.

Assumption IV. That cross sections of the beam which are plane and at right angles to the axis of the beam before bending, remain plane and normal to the axis after bending. That is, if the lines AB and CD in Fig. 169 represent such plane sections of the beam before bending, then the lines $A'B'$ and $C'D'$ may similarly be taken to represent the corresponding plane sections in the bent beam, their surfaces being normal to the curve which the axis of the beam has assumed. This is known as "Bernoulli's Assumption", after the eighteenth century mathematician who enunciated it.

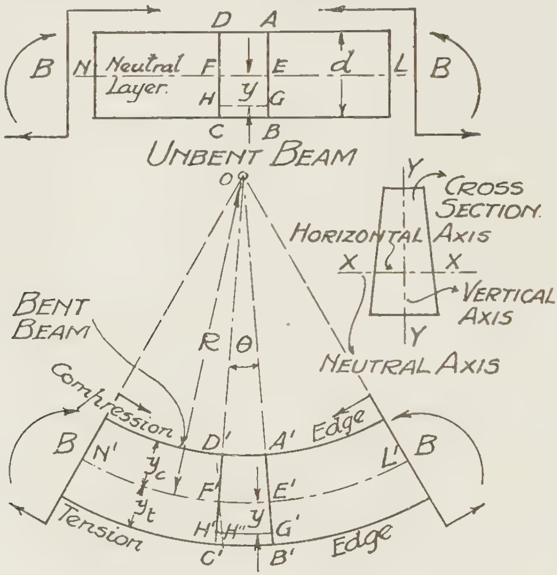


Fig. 169. Simple bending.

117. Relation between stress and the shape of the bent beam. Keeping in mind the assumptions just stated, let us consider two plane sections, represented by AB and CD on the drawing of the unbent beam, which are at right angles to the axis of the beam and which are *assumed to be a very short distance apart.* By Assumption IV these plane sections remain plane after bending and may be represented in the second figure by the lines $A'B'$ and $C'D'$. From this figure it will be seen that the upper surface or layer of the beam has been shortened or compressed, the length $A'D'$ being less than AD, while at the same time the bottom layer has been extended. It will also be seen that intermediate layers will range from one extreme to the other so that some layer FE will have been neither extended nor shortened, that is *the layer will*

not have been strained longitudinally, the length of $E'F'$ being the same as the original length EF. The layer will of course have bent to follow the curvature of the beam. The layer or surface represented by the line EF—which continues (see line NL) throughout the length of the beam—is known as the **Neutral Layer** or **Surface.** The line X–X on the section of the beam through which this plane passes is known as the **Neutral Axis**; see Fig. 169.

Now let the section planes represented by $A'B'$ and $C'D'$ intersect in some line, of which the point O is the vertical trace in the plane containing the vertical axis of the beam, and let the angle so formed be θ, and the distance from O to the neutral surface $E'F'$ be R. This distance R may be taken to be the "radius of curvature" of the curve to which the short length FE has been bent. The length R will usually be considerable, the actual distortion of the beam being small.

Let y be the distance between the neutral layer FE and some other layer HG which is nearer to the tension edge. Then the original length of HG is equal to FE, and also to $F'E'$, since the layer $F'E'$ is not strained. ......(a)

The strained length of HG is given by $H'G'$. If then $F'H''$ be drawn through F' and parallel to $OE'G'$, then $H''G'$ will evidently be equal to $F'E'$, that is to FE, that is to HG (see (a) above), which is the original length of HG.

It follows that $H'H''$ represents the *total strain* on HG. ......(b)

Now since the figures $OF'E'$ and $F'H'H''$ are similar figures, corresponding sides will bear the same ratio to each other, so that

$$\frac{y}{R} = \frac{H'H''}{F'E'} = \frac{H'H''}{HG} \qquad \text{(see (a) above)}$$

$$= \frac{\text{total strain in } HG}{\text{original length}} = \text{unit strain in } HG. \text{ ......(c)}$$

But, from (iv) para. 106, we have

$$E = \frac{\text{stress}}{\text{strain}}, \quad \text{or} \quad \text{strain} = \frac{\text{stress}}{E}.$$

Then if f be the *flexural stress* or *stress due to bending* in the layer HG, and E is the modulus of elasticity for the material of the beam, we have

$$\text{unit strain in } HG = \frac{f}{E}. \qquad \text{......(d)}$$

Then, combining (c) and (d), we have $f/E = y/R$,

or
$$\frac{f}{y} = \frac{E}{R}, \qquad \text{......(i)}$$

which is the first important result in this discussion. (Note that y may be measured either above or below the neutral layer.)

118. The Distribution of Stress. By Assumption IV the lines $A'B'$ and $C'D'$ in Fig. 169 are straight lines, and the strain in each layer is therefore proportional to the distance of each layer from the neutral layer, that is to the distance y. This is true both on the tension and on the compression side of the neutral layer. Hence if we draw a diagram to show the distribution of stress between layers, as in Fig. 170, on which f_t and f_c represent the flexural stresses at the tension and compression edges respectively, *the lines* bE *and* aE *must be straight lines, since the strain, and therefore the stress, in any layer is proportional to its distance from the neutral layer.*

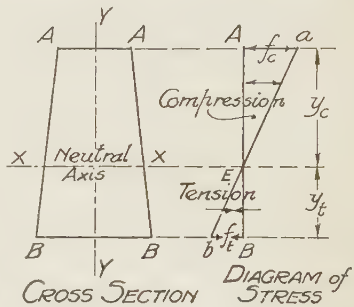


Fig. 170. Distribution of stress.

Hence we have that, **the longitudinal strains and also the stresses in the layers of a loaded beam, are proportional to the distances of those layers from the neutral layer.**

Again, since by Assumption II, we have that E is the same in tension and compression, it follows that stress and strain are similarly related in tension and in compression, and the inclinations of the lines bE and aE to the line AEB in Fig. 170 must be the same. Hence *the line* aEb *must be one continuous straight line.*

Further, it follows from the last statement that, if y_c and y_t be the distances from the neutral layer to the compression and tension edges respectively, then

$$\frac{f_c}{y_c} = \frac{f_t}{y_t}. \qquad \text{......(a)}$$

Also, at this section in the beam, f_c and f_t are clearly the maximum flexural stresses in compression and tension respectively.

The greatest value of the relation expressed by (i), para. 117 ($f/y = E/R$), lies not in its immediate practical value to us, but in the fact that an important and essential step in building up the theory of bending has been established. The following example will serve to show how the various units must be consistent with each other, and will emphasise just what are the limitations of this expression.

Example. *Calculate the radius of curvature* (R) *to which a mild steel bar,* 1 *inch in diameter* (d), *may be bent so that the maximum stress* (f) *in the bar reaches but does not exceed* 30,000 *lbs. per sq. in. The value of* E *may be taken at* 30,000,000 *lbs. per sq. in.*

In this case since the section is symmetrical the neutral axis will divide it into two equal parts, so that $y_c = y_t = d/2 = 0{\cdot}5$ in.

(**Note.** All quantities being in terms of inches and lbs. we should expect the answer to be given in the same terms.)

From para. 117 (i) we have

$$\frac{f}{y} = \frac{E}{R},$$

or

$$R = \frac{Ey}{f} = \frac{30{,}000{,}000 \times 0{\cdot}5}{30{,}000}$$

$$= 500 \text{ ins}$$

119. Position of the neutral axis. The foregoing example illustrates the relations included in expression (i), viz. $f/y = E/R$, but does not enable the stresses to be related to the bending moment (B), or to the dimensions of the beam, other than the depth in the case of a symmetrical section. Before we can extend the expression an important step is necessary, and that is to find the position which the neutral axis must occupy on the section of a beam.

In Fig. 171 is given a pictorial representation of the type of beam section which we are here considering, together with a graphical representation of the manner in which the stresses are distributed over such a section. We shall call this a "Distribution-of-stress Diagram". From this it will be clear that, while the stresses vary uniformly in the layers between the neutral layer and the top and bottom surfaces of the beam, the stress will be uniform in each layer across the breadth of the beam.

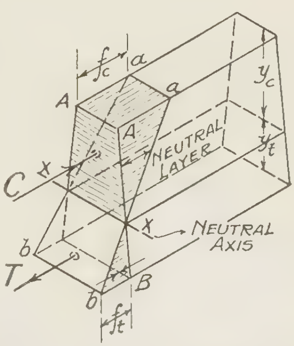


Fig. 171. Diagram of distribution of stress across a beam section.

On Fig. 171 the single large forces C and T may be taken to represent the total of all the compressive forces and all the tensile forces respectively which may be acting on the section. (For the present we need not concern ourselves with the exact magnitude or location of these two forces.)

In Fig. 172 is given a drawing of the section AB together with the accompanying diagram of stress, the terms used having the same meanings as in the preceding figures.

Let us consider a small element of area a on the section, situated at the distance y from the neutral axis X–X. If the intensity of stress acting on this area be f, then the total force acting on this small area a will be $a \times f$.

But from the figure we see that, by similar triangles,

$$\frac{f}{y} = \frac{f_c}{y_c},$$

whence $f = \left(\frac{f_c}{y_c} \times y\right)$.

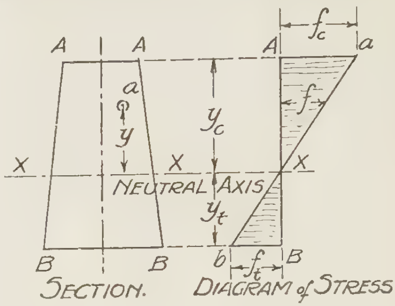


Fig. 172. Position of the neutral axis.

Therefore the total force on the element $a = a \times f$

$$= a\left(\frac{f_c}{y_c} \times y\right), \qquad \text{......(a)}$$

and the total compressive force C, on the area above the neutral axis X–X, which may be found by obtaining the sum of all the forces on these small units of area, may be found from

$$C = \Sigma\left[a\left(\frac{f_c}{y_c} \times y\right)\right],$$

which, since f_c/y_c is a constant, gives

$$C = \frac{f_c}{y_c}\Sigma\,(a \times y).$$

But the quantity $\Sigma\,(a \times y)$ is the first moment of the area about X–X, the neutral axis; see (iii) para. 15. Hence we have that,

total compressive force on the area above X–X

$$= C = \frac{f_c}{y_c} \times \text{(first moment of this area about } X\text{–}X\text{)}. \quad \text{...(b)}$$

Similarly, if we consider the force acting on the area below X–X, we get

total tensile force T, on the area below X–X,

$$= T = \frac{f_t}{y_t} \times \text{(first moment of this area about } X\text{–}X\text{)}. \quad \text{...(c)}$$

Now we know that, since this is a case of simple bending, there is no shear force acting across this section, and the external forces reduce to a couple equal to the bending moment (B) at this section. Then, for equilibrium, this bending moment must be exactly balanced by the internal resisting couple, i.e. the moment of resistance (R), or $B = R$; see para. 73.

The internal resisting couple is clearly formed by the two forces C and T, that is by the two forces defined in (b) and (c) above, and since these are the only two horizontal forces acting across the section, and they act in opposite directions, they must be equal to each other. Hence, giving due regard to sign, $T + C = 0$, or $\Sigma H = 0$, see para. 10.

Then, substituting from (b) and (c) above, we have

$$\frac{f_c}{y_c} \times (\text{first moment of area above } X\text{–}X)$$
$$+ \frac{f_t}{y_t} \times (\text{first moment of area below } X\text{–}X) = 0,$$

but, as we have seen, para. 118 (a),

$$\frac{f_c}{y_c} = \frac{f_t}{y_t};$$

hence, dividing through by either quantity, we have

$$(\text{first moment of area above } X\text{–}X)$$
$$+ (\text{first moment of area below } X\text{–}X) = 0.$$

That is, the total first moment of the section about X–X is equal to zero. But we saw in para. 15 (v), that the first moment of an area could only be zero about an axis which passed through the centroid of that area, hence we have that,

In simple bending the neutral axis passes through the centroid of the section.

120. Moment of Inertia (I) and Moment of Resistance (R). We have already seen that the external bending moment (B) = internal moment of resistance (R), see para. 73*, so that it only remains for us to relate the latter term to those representing the stresses acting in the strained beam and to the dimensions of the section.

As we have already shown, the total force carried by a small area a, see Fig. 172, which is situated at a distance y from the neutral axis X–X, is given by $a\,(f_c/y_c \times y)$; see (a), para. 119.

The moment of this small force about X–X will be given by

$$\left(a \times \frac{f_c}{y_c} \times y\right) y,$$

which

$$= a \times \frac{f_c}{y_c} \times y^2.$$

* The term R is commonly used to represent the moment of resistance; we have also used it to represent the radius of curvature. It is not often that these two terms occur together in the same expression. To avoid confusion, however, since the moment of resistance (R) = the external bending moment (B), the latter term (B) will be used instead of R wherever possible.

Now the total moment of all these forces over the *whole area* of the section, which must evidently be equal to the moment of resistance (R), is given by

$$\text{Moment of resistance } (R) = \Sigma\,(a \times f_c/y_c \times y^2),$$

which, since f_c/y_c is a constant ratio,

$$= \frac{f_c}{y_c} \times \Sigma\,(a \times y^2). \qquad \text{......(a)}$$

Moment of Inertia. The expression $(a \times y^2)$ is known as the "Second Moment of the small element of area a about the axis X–X", the Second Moment of *a very small area* being the product of the area into the square of its distance from some particular axis.

The quantity denoted by $\Sigma\,(a \times y^2)$ should therefore give *the sum of all the elements of area* (a), *multiplied by the squares of their distances* (y) *from the axis* X–X. Since, in this case, the axis X-X is the neutral axis of the area, we thus obtain "the Second Moment of the whole area of the section about the neutral axis".

The quantity $\Sigma\,(a \times y^2)$ is usually referred to as the **Moment of Inertia of the Section**, and is denoted by the letter I.*

As will presently be explained (see Chap. XV), it may be necessary to find the moment of inertia of an area about other axes than the neutral axis.

Moment of Resistance. Substituting I for the quantity $\Sigma\,(a \times y^2)$ in expression (a) above, and *remembering that the ratios* f_c/y_c *and* f_t/y_t *are identical if* E *has the same value in compression and tension*, we have

$$\textbf{Moment of resistance} = \boldsymbol{R} = \boldsymbol{B} = \frac{f_c}{y_c}\,I = \frac{f_t}{y_t}\,I. \qquad \text{......(ii)}$$

We may put expression (ii) in a more general form as

$$B = \frac{f}{y}\,I,$$

where f is the stress at any distance y from the neutral axis, from which we have

$$\frac{f}{y} = \frac{B}{I}. \qquad \text{......(iii)}$$

If we now combine this expression with (i) in para. 117, viz. $f/y = E/R$ (where R is the radius of curvature), we have an important and complete statement of the relations between the

* The term "moment of inertia" is borrowed from Dynamics and is not properly applicable in this case, but it is convenient and in common use and will therefore be retained.

stresses, strains and dimensions of a beam and the external moment (B) applied thereto; thus

$$\frac{f}{y} = \frac{B}{I} = \frac{E}{R}. \qquad \text{......(iv)}$$

121. Modulus of the section. The expression (ii) in the preceding paragraph can be arranged as follows:

$$B = f_c\left(\frac{I}{y_c}\right) = f_t\left(\frac{I}{y_t}\right).$$

The quantities within the brackets depend entirely upon the dimensions and shape of the cross section of the beam and are known as the **Compression Modulus** of the section (I/y_c) and the **Tension Modulus** of the section (I/y_t) respectively.

The letter Z is generally used to denote the Section Modulus, hence the above expression may be written:

$$B = f_c Z_c = f_t Z_t, \qquad \text{......(v)}$$

in which $$Z_c = \left(\frac{I}{y_c}\right) \text{ and } Z_t = \left(\frac{I}{y_t}\right).$$

In these expressions f_c and f_t stand for the stresses at the compression and tension edges respectively.

In those cases where the beam section is symmetrical about the neutral axis (see later examples) then $y_c = y_t$ and $f_c = f_t$, and we can put the expression in the general form

$$B = fZ, \qquad \text{......(vi)}$$

where f is the stress at either the compression or the tension edge of the beam, and Z is the section modulus (there being only one value since the section is symmetrical).

Since there are a number of practical problems which resolve themselves into finding a suitable beam section, when the external bending moment (B) and the maximum working stress (f) are given or can be found, expression (vi) is of very general use.

122. Examples. (1) *Find the value of the section modulus* (Z) *for a rectangular beam section which has to resist a bending moment* (B) *of* 288,000 *lb. ins., the stress* (f) *in the outer layers of the beam being limited to* 16,000 *lb. per sq. in.*

In this case, since the neutral axis will be at the centre of the depth of the beam section, y_c will equal y_t, and we may therefore use the expression

$$B = fZ.$$

Hence we have $$288{,}000 = 16{,}000 \times Z,$$

whence
$$Z = \frac{288{,}000}{16{,}000}$$
$$= 18 \text{ inch units}^3.*$$

(2) *If the beam in the preceding question has to be 8 ins. deep, what should be the value of the moment of inertia* (I)?

We already know that $Z = I/y$, where y is in this case half the depth of the beam, that is 4 ins.

Hence
$$I = Z \times y$$
$$= 18 \times 4$$
$$= 72 \text{ inch units}^4.*$$

(3) *If the beam described in Questions* 1 *and* 2 *is used to span an opening of* 10 *ft. and to carry a central load of* 4 *tons, see Fig.* 173, *what stresses would be induced at the centre of the beam* (a) *in the upper and lower layers, and* (b) *in the layers situated* 1 *in. from these surfaces?*

We must first find the bending moment (B).

Here
$$B = \frac{WL}{4}; \text{ see para. 83,}$$
$$= \frac{(4 \times 2240) \times (10 \times 12)}{4} \text{ lb. ins.}$$
$$= 268{,}800 \text{ lb. ins.}$$

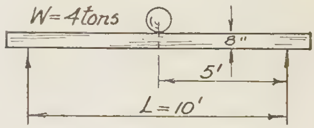


Fig. 173.

Now from the expression $f/y = B/I$ we have that the stress at the outer edge, where y is 4 ins., is given by
$$f = \frac{B \times y}{I}$$
$$= \frac{268{,}800 \times 4}{72}$$
$$= 15{,}000 \text{ lb. per sq. in. (approx.).}$$

Similarly the stress 1 in. from the outer surfaces, where y is 3 ins., is given by
$$f = \frac{268{,}800 \times 3}{72}$$
$$= 11{,}200 \text{ lb. per sq. in. (approx.)}$$

(Obviously we could have proceeded more directly because, since the stress is proportional to the distances of the fibres from the neutral axis, the second stress must be just three-quarters of the first.)

(4) *Given that* E *for the material of the beam described in the above Example is* 30,000,000 *lb. sq. ins., find the strain in the outer layers of the beam at the mid-section in the case given in Example* 3.
$$E = \frac{\text{stress}}{\text{strain}}.$$

* The significance and meaning of these units are explained in the next chapter.

Hence
$$\text{Strain} = \frac{\text{stress}}{E}$$
$$= \frac{15{,}000}{30{,}000{,}000}$$
$$= 0{\cdot}0005 \text{ inch, per inch of length.}$$

123. Simple or pure bending and ordinary bending. If the cases of beam loading dealt with in Chap. IX are reviewed, it will be seen that in practice cases of simple bending are comparatively rare and that, in the majority of cases, bending moment at any section is accompanied by shear force at the same section. The effect of this latter force on the distribution of stresses in the beam will presently be investigated in a simple manner (see Chap. XV), but it is possible to show at once, that the shear force is not usually a serious factor at that section at which the values are taken from which to calculate the size of the beam.

In practice, beams are usually of uniform section throughout, and the sizes of that section are usually designed so as to resist the maximum bending moment. Now, as we have already seen, the shear force is zero at those sections at which the bending moment passes through a maximum value (see Chap. X), so that at those sections the theory which we have outlined cannot be seriously at fault. The relations given above are in constant use by designers, and give results for ordinary cases which are not markedly different from those obtained by other methods, which may be theoretically more exact.

124. Experiment. *A complete bending test.* The following test has been devised so as to include a demonstration of the truth of each of the more important statements, contained in this and the succeeding chapters, concerning the conditions set up in the material of a loaded beam. The test is neither difficult nor lengthy, and does not require a high-powered testing machine. (So large a span as 8 ft., the one adopted in this case, is not absolutely essential, so that the test may be carried out on a machine only capable of accommodating a smaller span.) The various calculations are given below and in the following paras.: 113 (to find the value of E), 126 and 127 (to find the value of Z and I), 126 (to find the stresses acting in the beam), 149 (to find the deflection), 144 (to find the slope).

Beam and loading. A mild steel beam was used, the section being $b = 2$ ins., $d = 4$ ins. (The value of Z was $5{\cdot}33$ inch units, and the value of I was $10{\cdot}66$ inch units; see para. 126.)

The beam was carried on rounded supports, see Fig. 174, over a span of 8 ft. and, by means of the arrangement shown in Fig. 174, two equal loads of W lbs. (each equal to half the load transmitted from the testing machine head) were applied at points C and D 2 ft. apart, and 3 ft. from each end A and B.

For any pair of loads it will thus be seen that there was a constant bending moment of ($36\ W$) lb. ins. between the points C and D; see also Fig. 168.

Strain. The strain was measured at four different layers spaced as shown in Fig. 175 (A). It will be noted that these layers are symmetrically placed about the neutral axis.

Measurement of strain. The measurement of the strain in these layers was carried out by means of the Extensometer shown in Fig. 175 (B). This particular extensometer was intended for use on fairly large specimens in tension and in compression, the "gauge length" (AB) of 10 ins. is therefore somewhat larger than usual. The advantage of this particular type is that the extensometer may be easily applied and held to the specimen in various positions. In this case it was held up to the specimen by means of stout rubber bands passing over the two pins M and N and so round the specimen, the weight of the extensometer being taken by blocks of wood placed beneath it.

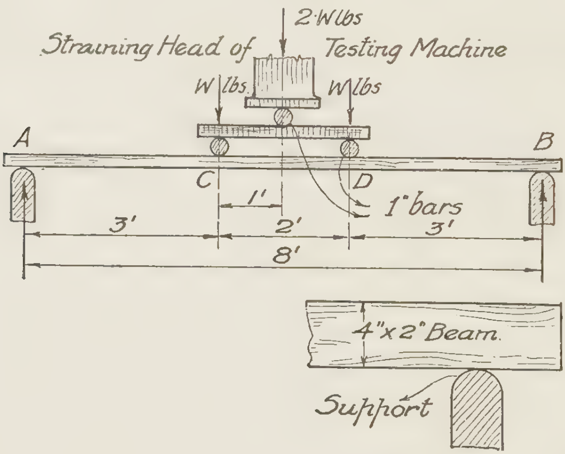


Fig. 174. Test beam and loading.

The principle of construction should be easily understood. The two points A and B being placed in punch holes exactly 10 ins. apart, any strain occurring in this length will result in the movement of the pivoted pin A relative to the fixed pin B. This movement actuates the lever AD, which is pivoted at C so that at point D the movement of A is multiplied 10 times.

The movement of D is shown on a dial micrometer capable of reading to 0·001 in. It will thus be seen that movements of 1/10,000 in. between A and B will be indicated on the dial of the micrometer.

Graphs. The strain was measured at each layer for changes of the load W of 500 lbs., both as the load was being increased and also as it was being reduced.

From these results the graphs shown in Fig. 176 were drawn, the continuous lines showing the measured strains in the corresponding layers, A–A, B–B, etc. in the beam.

Comparison with calculated strains. Using the value of E (30,000,000 lbs. per sq. in.) obtained for the material of this beam (see para. 113), the

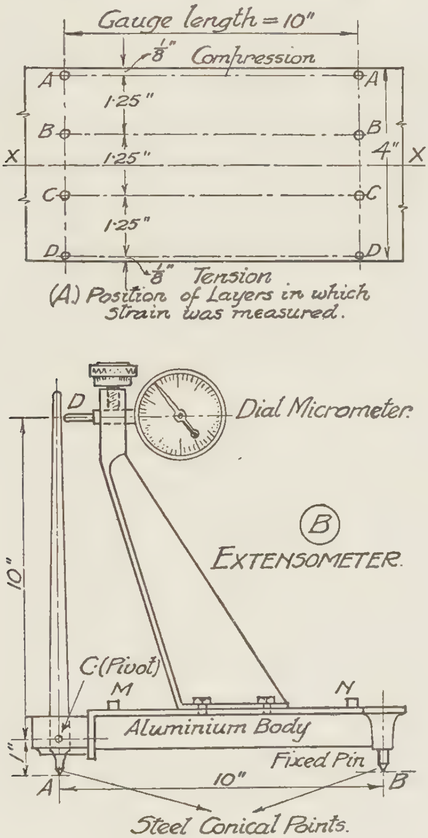


Fig. 175. Measurement of strain in a beam.

strains in these layers have been computed from the calculated stresses given in para. 126; from these values the dotted graphs shown in Fig. 176 have been plotted. The two experimental graphs (tensile and compressive strains) practically coincide in each case, and only depart slightly from the calculated graphs. Allowing for experimental errors, these results confirm our statements regarding strain in a loaded beam. The value of E is also shown, by the inclination of these lines, to be the same for compression and tension, and to have the same value as used in calculating the strains.

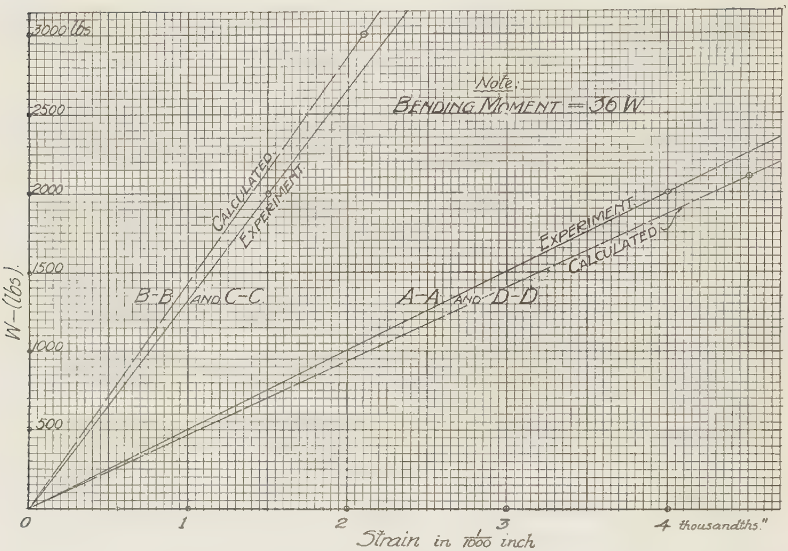


Fig. 176. Measured and calculated strains in a loaded beam.

(In para. 126 the stresses at different layers are similarly compared, as obtained by calculation from the bending moment (B), and also from the measurement of strain in the layers.)

Finally in Fig. 177 is shown a diagram of the depth of the beam, drawn to scale, on which has been plotted the strains in the four layers at a given bending moment (B) of (3000×36) or 108,000 lb. ins.; compressive strains being plotted above X–X and to the right of Y–Y,

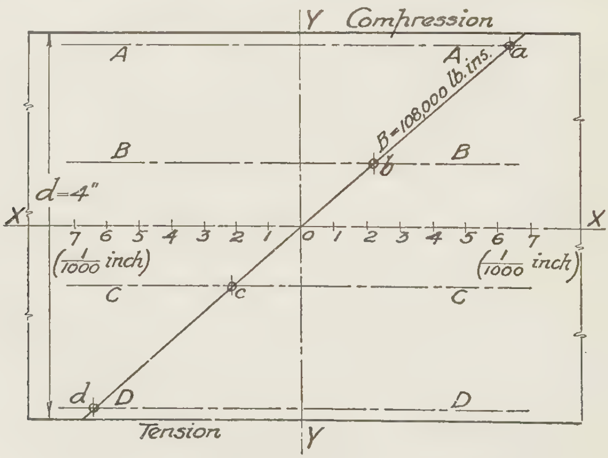


Fig. 177. Strain at different layers in a loaded beam.

and tensile strains below and to the left. The line drawn through the points so found (a, b, c and d) will be found to be a straight continuous line (similarly for other values of B), thus demonstrating that plane sections before bending remain plane after bending, the strains being proportional to the distances of the layers from the neutral axis.

(**Note.** It is important to remember that the whole of the discussion given above is based on the assumption that *the material is not strained beyond the elastic limit.*)

Problems XIII

1. If the beam RS in Fig. 168 is of I-section and 8 ins. deep, calculate the radius of curvature (R) to which the beam will be bent between the points M and N, when the stress in the outer layers is 16,000 lbs. per sq. in. and $E = 30{,}000{,}000$ lbs. per sq. in.

2. What will be the stresses in the layers 1 in. and 3 ins. respectively from the upper or lower surface of the beam in Prob. 1?

3. If the moment of inertia (I) for the beam in Prob. 1 be 56 inch units^4, calculate the value of the bending moment (B) which produces a stress of 16,000 lbs. per sq. in. in the outer layers.

4. Calculate the total strain over a gauge length of 10 ins. at the outer edge and at 1 in. and 3 ins. respectively from the outer edge, in the beam in Prob. 1.

5. Calculate the safe uniformly distributed load which the beam in Prob. 3 would carry over a span of 12 ft. if the bending stresses are limited to $7\frac{1}{2}$ tons per sq. in.

6. A cast iron beam has a section similar to that shown in Fig. E, Probs. III. (a) If the distance from the neutral axis X-X to the compression edge (y_c) is 6·5 ins. while the distance to the tension edge (y_t) is 3·5 ins., find the value of Z_c and Z_t respectively if the moment of inertia is 200 inch units^4. (b) If the tensile stress is limited to $1\frac{1}{2}$ tons per sq. in., calculate the maximum moment of resistance (R) for the section. (c) What would be the maximum compressive stress when the tensile stress was $1\frac{1}{2}$ tons per sq. in.?

CHAPTER XIV

THE COMPARISON AND DESIGN OF SOLID BEAM SECTIONS

125. To find the value of the modulus of section (Z) for a rectangular section. Let a beam be subjected to a bending moment (B), the section of the beam being represented by the rectangle $AABB$ in Fig. 178. The neutral axis X–X will divide the rectangle into two equal portions as shown, so that the stress diagram will also be symmetrical, f_c and f_t being equal to f, the maximum stress induced (tension or compression) by the bending moment B.

Since the stress on the compressive area $AAXX$ varies uniformly from zero at the neutral axis to f at compression edge, the average stress will be $f/2$. Then the total compressive force acting on the area $AAXX$ = (area) × (average stress)

$$= \left(b \times \frac{d}{2}\right)\frac{f}{2} = f\left(\frac{bd}{4}\right),$$

which expresses the magnitude of the compressive force C, or

$$C = f\left(\frac{bd}{4}\right). \qquad \text{......(a)}$$

In the same way we may show that the total force (T) acting below X–X has also the same value, hence

$$T = f\left(\frac{bd}{4}\right). \qquad \text{......(b)}$$

As before, the distribution of stress across the section may be represented by a pictorial diagram, from which it is seen that, since the stress acting across any layer parallel to X–X is uniform, the distribution of stress diagram consists of two wedge-shaped figures or "solids", see Fig. 178. It is not difficult to show that the magnitude of the forces C and T is in each case equal to the "volume" of the "solid" and acts at its "centre of gravity", that is at a distance of 2/3 ($d/2$), or $d/3$ from X–X in each case; see Fig. 178.

The distance between the lines of action of C and T is thus $2d/3$, and, since C and T are opposite and equal forces and form a couple, then the moment of this couple, that is the moment of resistance (R), $= C \times 2d/3 = T \times 2d/3$.

Then, substituting for C (or T) from (a) or (b) above, we have

$$\text{Moment of resistance} = R = f\left(\frac{bd}{4}\right)\frac{2d}{3}$$

$$= f\left(\frac{bd^2}{6}\right). \qquad \text{......(c)}$$

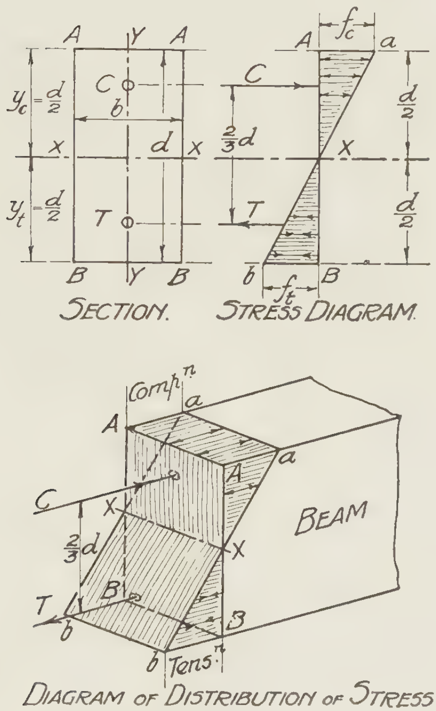


Fig. 178. Rectangular beam section.

Now from expression (vi), Chap. XIII, we have $R = B = fZ$, and comparing this expression with (c) above it is evident, by similarity, that the quantity in brackets expresses the value of the section modulus (Z) in the case of a rectangle. Then

$$\text{The modulus of section } (Z) \text{ for a rectangle} = \frac{bd^2}{6} \text{.......(i)}$$

Units. Since the value of Z is obtained from ($b \times d \times d/6$), if b and d be measured in inches, then the answer will be in (inch units)^3 or "inch units^3".

Modulus Figure. If, as is shown in Fig. 178 A, the rectangle $CEFG$, representing the section of a beam, be divided by drawing the two

diagonals CF and GE, then the two shaded triangles so produced will have areas each equal to

$$\left(b \times \frac{d}{2} \times \frac{1}{2}\right) \text{ or to } \left(\frac{bd}{4}\right).$$

If we imagine a uniform stress equal to f, the maximum stress acting in the beam, to act over the whole of the shaded area, compressive stress acting above X–X, and tensile stress below, as indicated in the special "diagram of uniform stress" shown in Fig. 178 A, then the total compressive force acting above X–$X = f\,(bd/4) = C$. Similarly the tensile force $= f\,(bd/4) = T$.

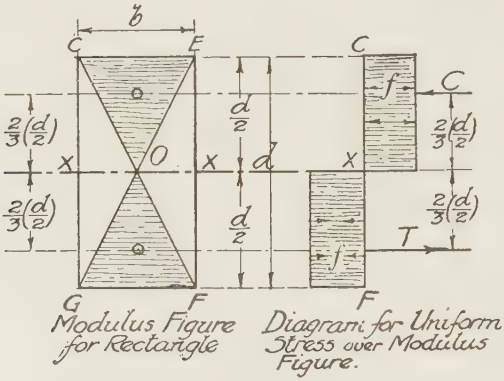


Fig. 178 A.

Again these forces will act at the C.G.'s of the triangles, or $d/3$ from X–X. The forces C and T are thus $2d/3$ apart, so that the magnitude of the couple thus formed

$$= f\left(\frac{bd}{4}\right)\frac{2d}{3} = f\left(\frac{bd^2}{6}\right).$$

This is identical with expression (c) above for the moment of resistance. The figure $CEOGF$, over which the imaginary uniform stress f acts, is known as the **Modulus Figure** for this section. Graphical methods are available for drawing the modulus figure both for simple and complex sections, and the method is occasionally used for finding the value of Z in cases where a direct solution by calculation is not easy.

126. Experiment. *To find the stresses acting in the beam tested in the Complete Bending Test described in para.* 124.

In the table given below are tabulated values obtained in the above test. Column 1 gives the bending moments acting on the portion CD, see Fig. 174, and corresponding to successive increases of W of 500 lbs. Columns 2 and 3 give the measured strains (in thousandths of an inch) in the two layers B–B and D–D. In Columns 4 and 5 the corresponding stresses have been calculated using the value of E (30,000,000 lbs. per sq. in.), from the expression, stress $= E \times$ strain; see para. 106.

In Columns 6 and 7 are given the stresses in the same layers as calculated from the expression $B = fZ$, as follows:

The value of Z in this case can be found from the expression (i) in this chapter

$$Z = \frac{bd^2}{6} = \frac{2 \times 4^2}{6} = 5{\cdot}33 \text{ inch units}^3$$

(the dimensions of the beam section being $b = 2$ ins. and $d = 4$ ins.).

Then the stress at the compression or tension edge will

$$= f = \frac{B}{Z} = \frac{B}{5{\cdot}33},$$

from which we are able to find the stresses at the outer edges for each value of B. Now the outer edges are 2 ins. from the neutral axis. The stresses in the other layers will be proportional to their distances from the neutral axis. The distances from the neutral axis to these layers are 1·875 (A–A and D–D) and 0·625 (B–B and C–C) respectively.

It is then only a simple exercise in proportion, to calculate the stresses acting in these layers for the corresponding stresses at the outer edges. These values for B–B and D–D are shown in Columns 6 and 7.

Stress and Strain in a Loaded Beam, see Experiment, para. 124.

Bending moment (lb. ins.)	Measured strain (unit 0·001 in.)		Stress, from ($E \times$ strain) (lbs. per sq. in.)		Stress, from $f = \frac{B}{Z}$ (lbs. per sq. in.)	
	Layer B–B	Layer D–D	Layer B–B	Layer D–D	Layer B–B	Layer D–D
(1)	(2)	(3)	(4)	(5)	(6)	(7)
18,000	0·375	1·065	1,125	3,225	1,055	3,165
36,000	0·75	2·15	2,250	6,450	2,145	6,430
54,000	1·125	3·2	3,375	9,600	3,165	9,495
72,000	1·5	4·275	4,500	12,825	4,287	12,860
90,000	1·875	5·36	5,625	16,080	5,275	15,820
108,000	2·25	6·4	6,750	19,200	6,330	18,990

The closeness of the values given in Columns 4 and 6 and also in 5 and 7 is sufficient to confirm the statements made above, regarding the relations between stress and strain in loaded beams, and between the external bending moment (B) and the internal moment of resistance (R), which as we have seen is equal to fZ.

127. To find the value of I from a known value of Z. If we are given the value of Z for any particular symmetrical section, we may find the value of I for the same section from the relation $Z = I/y$ (see para. 121), from which we have

$$I = y \times Z. \qquad \text{......(ii)}$$

For example, in the case of the rectangle, $y = d/2$ and $Z = bd^2/6$; then

$$I = y \times Z = \frac{d}{2} \times \frac{bd^2}{6} = \frac{bd^3}{12}.$$

This gives the value of the moment of inertia for a rectangle about an axis through its centroid, or

The moment of inertia of a rectangle about an axis through its centroid is given by

$$\frac{bd^3}{12}. \qquad \text{......(iii)}$$

Units. Since I is found from $(b \times d \times d \times d/12)$ if each of the quantities b and d are in inches, then the answer will be in "inch units[4]".

Example. *Find the value of* I *for the section of the beam in para.* 124, *where* b = 2 *and* d = 4 *ins.*

$$I = \frac{bd^3}{12} = \frac{2 \times 4^3}{12} = 10\cdot66 \text{ inch units}^4.$$

128. To find the moment of inertia of a rectangle by means of the calculus. This is the most generally useful method for finding I for any section bounded by a geometrical outline.*

(*a*) To find the moment of inertia of a rectangle about one side. See Fig. 179 (A). Let $CEFG$ be a rectangle, of which we desire to know the moment of inertia about the axis S–S, passing along the side GF. We will refer to I about S–S as I_{SS}.

Taking a thin strip of thickness δy, at a distance y from S–S, then the moment of inertia of this strip about S–S

= (area of strip)

× (square of distance from S–S)

$= (b \times \delta y)\, y^2 = by^2\delta y.$

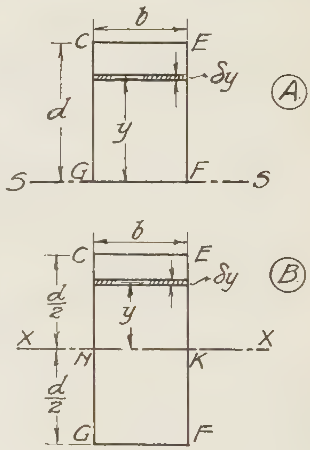


Fig. 179.

Then the moment of inertia of the whole figure about S–S will be given by the sum of the moments of inertia of all the small strips into which it may be divided, or

I_{SS} = moment of inertia of rectangle about S–S

$$= \int_{y=0}^{y=d} by^2\delta y = b\left[\frac{d^3}{3}\right]_0^d$$

$$= \frac{bd^3}{3}. \qquad \text{......(iv)}$$

* Those readers who are not familiar with the mathematical methods adopted in this paragraph may omit the paragraph.

(*b*) To find the moment of inertia of a rectangle about an axis through its centroid. See Fig. 179 (B). In this case the axis X–X divides the rectangle into two equal parts; hence we may find the moment of inertia by the method employed in (*a*) or, using expression (iv), find it by adding together the moment of inertia of the two rectangles of which the figure is composed. The moment of inertia of rectangle $CEKN$ about X–$X = \dfrac{b\,(\text{depth})^3}{3}$ (where breadth $= b$, and depth $= d/2$ in this figure)

$$= \frac{b\left(\frac{d}{2}\right)^3}{3} = \frac{bd^3}{24}.$$

Similarly for rectangle $NKFG$.

Adding these two quantities together to get the moment of inertia for the whole figure about X–X, we have

$$I_{XX} = \frac{bd^3}{12}, \text{ as in (iii) above.}$$

129. To find I for compound rectangular sections. The sections shown in Fig. 180 are called a "Box section" and an "**I**-section" respectively. For the sake of simplicity similar terms have been used to indicate corresponding dimensions in the two sections. If we consider the box section it should be obvious that its moment of inertia will be equal to the difference between I for the whole figure and I for the inner figure, the subtraction of which leaves the form of the box section. This must also be true for the **I**-section, since the area of the section and also the disposition of the material about the neutral axis X–X are the same; indeed, considered solely as beam sections, the two figures are identical. Then

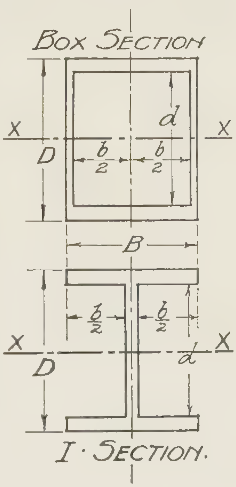


Fig. 180.

I_{XX} for **I**-section $= I_{XX}$ for box section

$$= (I \text{ for outer figure}) - (I \text{ for inner figure})$$

$$= \frac{BD^3}{12} - \frac{bd^3}{12}. \qquad \ldots\ldots(\text{v})$$

130. Circular and hollow circular sections. *The moment of inertia* (I_{XX}) *for a circular section* may be shown to

$$= \frac{\pi}{64} D^4, \qquad \ldots\ldots\text{(vi)}$$

where D is the diameter of the circle; see Fig. 181. It therefore follows that for *a hollow circular section*, such as that shown in Fig. 181,

$$I_{XX} = \frac{\pi}{64}(D^4 - d^4). \qquad \ldots\ldots\text{(vii)}$$

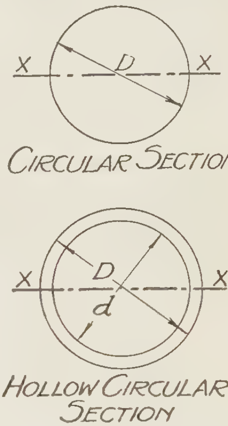


Fig. 181.

Example 1. *If in Fig.* 180 *the dimensions of the box section are as follows,* B = 10 *ins.,* b = 9 *ins.,* D = 12 *ins.,* and d = 10 *ins., find the value of* I.

$$I_{XX} = \frac{BD^3}{12} - \frac{bd^3}{12}$$

$$= \frac{10 \times 12 \times 12 \times 12}{12} - \frac{9 \times 10 \times 10 \times 10}{12}$$

$$= 1440 - 750$$

$$= 690 \text{ inch units}^4.$$

Example 2. *If in Fig.* 181 *the dimensions of the hollow circular section are* D = 10 *ins. and* d = 9 *ins., find the value of* I *about the neutral axis.*

Here
$$I_{XX} = \frac{\pi}{64}(D^4 - d^4)$$

$$= \frac{\pi}{6}(D^2 + d^2)(D^2 - d^2)$$

$$= \frac{\pi}{64}(181 - 19)$$

$$= 169 \text{ inch units}^4 \text{ (approx.)}.$$

131. Finding I for built-up sections and for sections which are not symmetrical about a horizontal axis. The Principle of Parallel Axes. In dealing with such sections as may be included under the heading to this paragraph the following principle, which is known as the Principle of Parallel Axes, is of great value. Let any figure, such as that shown in Fig. 182, have an area A and let its moment of inertia about its neutral axis X–X be I_{XX}. If now an additional axis Z–Z be taken, which is parallel to X–X and at a distance y from it, then it may be shown that, *the moment of inertia of the area* A *about the parallel axis* Z–Z *is equal to its own moment of inertia about its neutral axis* (I_{XX}) *plus the area* (A) *of the figure multiplied by the square of the distance* (y^2) *between the two axes.*

Thus, if I_{ZZ} be the moment of inertia of the figure about the axis Z–Z, we have

$$I_{ZZ} = I_{XX} + Ay^2. \qquad \ldots\ldots\text{(viii)}$$

The following worked examples will suffice to show how the principle may be applied in various cases.

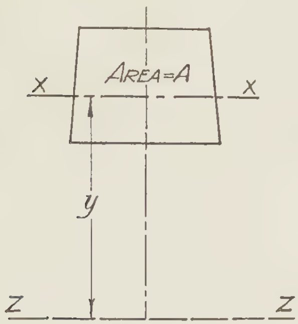


Fig. 182. The principle of parallel axes.

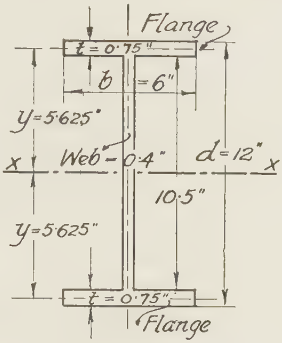


Fig. 183. **I**-section.

Example 1. *To find* I *for an* **I**-*section using the principle of parallel axes.* See Fig. 183.

To find I_{XX} by this method, we consider the moment of inertia of each portion of the section about the neutral axis separately, adding the results together to obtain I_{XX} for the whole figure. There will be three separate portions as follows:

(*a*) *The moment of inertia of the centre section or web which, using* $\dfrac{bd^3}{12}$,

$$= \frac{0{\cdot}4 \times 10{\cdot}5^3}{12} = 38{\cdot}5 \text{ inch units}^4.$$

(*b*) *The moment of inertia of the top and bottom flanges about their own neutral axes* (situated at the middle of the thickness in each case)

$$= 2\left(\frac{6 \times 0{\cdot}75^3}{12}\right) = 0{\cdot}421 \text{ inch units}^4.$$

(*c*) $(A \times y^2)$ *for the top and bottom flanges*

$$= 2\,(6 \times 0{\cdot}75 \times 5{\cdot}625^2) = 284{\cdot}76 \text{ inch units}^4.$$

(Note. The statements (*b*) and (*c*) are of course based on the principle of parallel axes.)

Then, adding (*a*), (*b*) and (*c*), we have

$$I_{XX} = 38{\cdot}5 + 0{\cdot}421 + 284{\cdot}76 = 324 \text{ inch units}^4 \text{ (approx.)}.$$

(Note. The value of I_{XX} may also be found by the method described in para. 129.)

Example 2. *Find I for the cast iron girder section shown in Fig.* 184.

The position of the centroid of this section was found in the worked example given in para. 15, where it was shown to lie on the centre line and 3·4 ins. from the bottom. This and the other dimensions necessary in the following calculations are indicated on the figure.

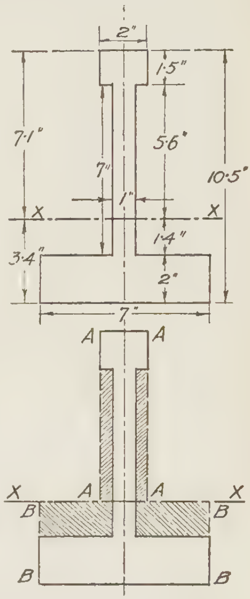


Fig. 184. Cast iron section.

The work may proceed along two lines, (*a*) by utilising the principle given in this paragraph, or (*b*) by means of a method similar to that used in finding I for box and I-sections. Since the method in the first case is exactly the same as in the example just worked—the only difference arising out of the necessity for dealing with each flange separately—only the outline is given here. The calculations involved have been included as a problem at the end of this chapter.

(*a*) *To find* I *by means of the Principle of Parallel Axes.* The work is divided into the following parts: (1) the value of I for the two portions of the web (about X–X), (2) the value of I for each flange about its own neutral axis, (3) the value of ($A \times d^2$) for each flange about XX. The value of I_{XX} for the whole section is then given by the sum of the quantities thus found.

(*b*) *To find* I *by the second method.* This method is frequently the easier to apply and is therefore worked out fully. The procedure is first to find the sum of the moments of inertia for the two enclosing rectangles AA and BB about X–X. This is then reduced by the value of the sum of I for the two pairs of shaded rectangles about X–X, the result giving the value of I_{XX} for the whole figure.

Moment of inertia of $AAAA$ about X–X (using $bd^3/3$)

$$= \frac{2 \times 7{\cdot}1^3}{3} = 238 \text{ inch units}^4 \text{ (approx.)}.$$

Moment of inertia of $BBBB$ about X–X

$$= \frac{7 \times 3{\cdot}4^3}{3} = 91{\cdot}6 \text{ inch units}^4 \text{ (approx.)}.$$

Moment of inertia of upper shaded rectangles about X–X

$$= 2\left(\frac{0{\cdot}5 \times 5{\cdot}6^3}{3}\right) = 58{\cdot}5 \text{ inch units}^4 \text{ (approx.)}.$$

Moment of inertia of lower shaded rectangles about X–X

$$= 2\left(\frac{3 \times 1{\cdot}4^3}{3}\right) = 5{\cdot}5 \text{ inch units}^4 \text{ (approx.)}.$$

Then I_{XX} for the whole figure

$$= 238 + 91{\cdot}6 - (58{\cdot}5 + 5{\cdot}5)$$
$$= 265{\cdot}6 \text{ inch units}^4 \text{ (approx.)}.$$

Example 3. *To find an expression for the moment of inertia of a triangle about its neutral axis.*

While this example should be omitted by those readers who are not able to follow the mathematical methods which are used, the results should be noted by all.

We shall first ascertain the moment of inertia of the triangle about an axis Z–Z, see Fig. 185, which passes through the apex A and is parallel to the base of the triangle. Taking an exceedingly thin strip, of a thickness δy, situated at a distance y from A and lying parallel to the base BC, then its area will be given by $(b_1 \times \delta y)$, where b_1 is the breadth of the figure at this level.

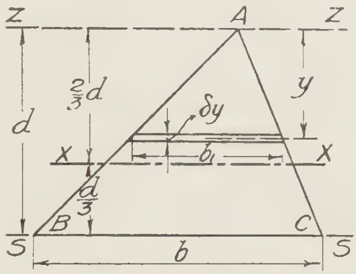


Fig. 185. I for a triangle.

Now, by similar triangles, we have

$$\frac{b_1}{b} = \frac{y}{d}, \text{ whence } b_1 = \frac{yb}{d}.$$

Substituting this value of b_1 in the expression giving the area, we have

$$\text{Area of strip} = \left(\frac{y \times b}{d}\right) \times (\delta y).$$

Then the moment of inertia of the strip about Z–Z

$$= \text{area} \times \text{distance}^2$$

$$= \left[\left(\frac{y \times b}{d}\right) \times (\delta y)\right] \times y^2 = \frac{by^3}{d}\,\delta y.$$

Then the moment of inertia of the whole figure about Z–Z will equal the sum of these small quantities, or

$$I_{ZZ} = \int_{y=0}^{y=d} \frac{by^3}{d}\,\delta y$$

$$= \frac{b}{d}\left[\frac{y^4}{4}\right]_0^d = \frac{b}{d} \times \frac{d^4}{4}$$

$$= \frac{bd^3}{4}. \qquad \text{......(ix)}$$

To find the moment of inertia of the triangle about the neutral axis X–X we employ the principle of parallel axes.

Since X–X passes through the centroid of the triangle it evidently lies at a distance of two-thirds d from A; hence, from

$$I_{ZZ} = I_{XX} + Ay^2, \text{ where } y = \left(\frac{2d}{3}\right),$$

we have
$$\frac{bd^3}{4} = I_{XX} + \left(\frac{bd}{2} \times \frac{4d^2}{9}\right),$$

or
$$I_{XX} = \frac{bd^3}{4} - \frac{2bd^3}{9} = \frac{bd^3}{36}. \qquad \text{......(x)}$$

The moment of inertia of a triangle about its base S–S is

$$I_{SS} = \frac{bd^3}{12}. \qquad \ldots\ldots\text{(xi)}$$

The proof of this is left as a problem for the reader; see Problems at the end of this chapter.

132. An approximate expression for the I and Z of an I-section. If the quantities (a) and (b) in Example 1, given in para. 131, be considered, it will be seen that they are relatively small compared with the quantity (c). This fact enables us to devise approximate expressions for I and Z, which are of considerable value where rough preliminary calculations are to be made. In these approximate expressions the quantities (a) and (b) are ignored, and (y) in the expression ($A \times y^2$) is replaced by ($d/2$), or half the total depth of the beam; see Fig. 186. Then

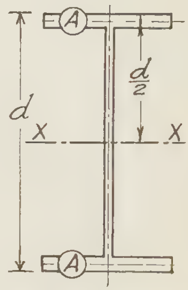


Fig. 186. Approximate moment of inertia for an I-section.

I (approximate) for I-section

$$= 2\,(\text{area of flange}) \times (\text{half depth of beam})^2$$

$$= 2A\left(\frac{d}{2}\right)^2 = \frac{Ad^2}{2}. \qquad \ldots\ldots\text{(xii)}$$

From this we may find an approximate expression for the section modulus Z of such a section.

Since $Z = I/y$, where $y = d/2$, we have

$$Z \text{ (approx.) for I-section} = \frac{I\,(\text{approx.})}{\frac{d}{2}}$$

$$= \frac{Ad^2}{2} \times \frac{2}{d} = Ad, \qquad \ldots\ldots\text{(xiii)}$$

or, in words, *the value of the section modulus* (Z) *for an* **I**-*section is approximately equal to the area of one flange multiplied by the depth of the beam.* (In practice the approximate expression for Z is more often used than that for I.)

Example. *Find the approximate expressions for* I *and* Z *for the* **I**-*section given in Example* 1 *in para.* 131.

Here $A = 6 \times 0{\cdot}75 = 4{\cdot}5$ sq. ins.,

$d = 12$ ins.

Hence $I_{XX}\,(\text{approx.}) = \dfrac{Ad^2}{2}$

$$= \frac{4{\cdot}5 \times 12 \times 12}{2} = 324 \text{ inch units.}$$

Also $$Z \text{ (approx.)} = Ad = 4{\cdot}5 \times 12$$
$$= 54 \text{ inch units.}$$

(The values in this case are thus seen to be identical with those found in Example 1, para. 131.)

133. The comparison and design of solid beam sections. It should now be clear that, apart from the question of shear stress and strain, with which we shall deal in the next chapter, the problem of designing beam sections resolves itself, in general terms, into one of applying the expression $f/y = B/I$ in one or other of its forms, including in particular the expression $B = fZ$. We shall deal at a later stage with certain practical points which must usually be observed in such work, but it will be useful to note at this stage the value of the terms I and Z as a means of comparing the strengths of two or more beams of given sections.

Taking a simple rectangular beam in the first instance, it will be seen, from the expression $B = fZ = fbd^2/6$, that *the strength of a beam of rectangular section varies directly as the square of its depth* (d), *and also directly as its breadth* (b); a statement which emphasises the importance of the vertical dimensions of a beam where strength is the main consideration.

If we remember that the quantity I is built up from the small quantities ($a \times y^2$), see para. 120—in which the significance of the term y is clearly indicated—it is possible to realise how much more effective is the box or **I**-section, as shown in Fig. 180, from the point of view of strength in bending, than one in which so much of the material is not placed near the upper and lower edges of the beam. It is on this fact that the special design of the box or **I**-section is based, in which the bulk of the material is placed near the upper and lower edges of the beam, where the flexural stresses f_c and f_t reach their greatest values.

The following examples will show how the principles dealt with in this chapter are applied in the solution of various practical problems.

Example 1. *In an experiment (see "Complete Bending Test", para.* 124) *a solid mild steel beam,* 4 *ins.* (d) *by* 2 *ins.* (b) *in section, was loaded as shown in Fig.* 187. *Using the dimensions indicated on the figure, determine the value of the two equal loads* W *(lbs.) so that the maximum stress in the beam just reaches* 20,000 *lbs. per sq. in.*

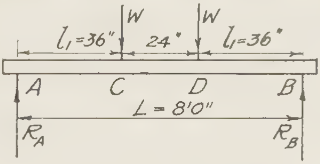


Fig. 187. Test beam loaded at two points.

Here the maximum value of the bending moment (B) is reached at both C and D and remains constant between those two points, or

$$B_c = B_d = R_A \times l_1$$
$$= W \times l_1 = W \times 36 \text{ lb. ins.}$$

Again $$I_{XX} = \frac{bd^3}{12} = \frac{2 \times 4^3}{12} = \frac{32}{3} \text{ inch units}^4.$$

Also $y = \frac{4}{2} = 2$ ins. and $f = 20{,}000$ lbs. per sq. in.

Then from $$\frac{B}{I} = \frac{f}{y},$$

see (iii), para. 120, we have
$$B = \frac{f \times I}{y},$$

or $$36W = 20{,}000 \times \frac{32}{3} \times \frac{1}{2},$$

whence $$W = 2963 \text{ lbs.}$$

Example 2. *If the maximum stress on the material of a timber beam is to be limited to 1000 lbs. per sq. in., find the maximum distributed load which a 9 in. by 2 in. timber joist would carry across a span of 12 ft. At what maximum spacing could these joists be placed if they are to form part of a floor which is to carry a load of 100 lbs. per foot of superficial area?*

In this case $B = wL^2/8$, where $L = 12$ ft. $= 144$ ins. and w is the load per inch of length of the joist. Also

$$f = 1000 \text{ lbs. per sq. in., and } Z = \frac{bd^2}{6} = \frac{2 \times 9 \times 9}{6} = 27 \text{ inch units}^3.$$

Then from $$B = \frac{wL^2}{8} = fZ,$$

we have $$\frac{w \times 144 \times 144}{8} = 1000 \times 27,$$

whence $$w = \frac{27{,}000 \times 8}{144 \times 144} = 10{\cdot}4 \text{ lbs. per inch linear,}$$

or $$124{\cdot}8 \text{ lbs. per foot linear.}$$

This value indicates that the floor would carry 124·8 lbs. *per square foot if the joists were spaced one foot apart*, but the floor is only required to carry 100 lbs. per square foot; hence the spacing may be wider.

$$\text{The spacing will} = 12 \times \frac{124{\cdot}8}{100}$$
$$= 14{\cdot}976, \text{ or say ``15 ins. centres''.}$$

Example 3. *Compare the strengths of two timber joists, both 8 ins. by 3 ins. in section, when one is placed as a beam resting on its edge and the other on its side.*

From the expression $B = fZ$ we see that, if f has the same value in two cases, then the strengths of the beams will be to each other as the value of Z in each case.

In the present instance the joist which is placed on edge will have

$$Z = \frac{3 \times 8 \times 8}{6} = 32 \text{ inch units}^3.$$

The joist placed on its side will have

$$Z = \frac{8 \times 3 \times 3}{6} = 12 \text{ inch units}^3.$$

Hence the strengths of the two joists will be as 32 to 12. This shows how important it is to place a beam of rectangular section so that the larger dimension of the section is vertical.

Example 4. *Cast iron is much weaker in tension than in compression and the strength of a cast iron girder is usually limited by the maximum safe tensile stress which the material can bear.*

Using the dimensions of the section of a cast iron girder shown in Fig. 184, *find what central load such a girder would carry over a span of* 8 *ft. if the stress on the tension edge must be limited to* $1\frac{1}{2}$ *tons per sq. in. Find the maximum compressive stress induced with this loading.*

Using the values given or found in Example 2, para. 131, we have

y_t = distance to tension edge from X–X = 3·4 ins.

y_c = distance to compression edge = 7·1 ins.

I_{XX} = 265·6 inch units^4.

Hence we have

$$Z_t = \frac{I}{y_t} = \frac{265 \cdot 6}{3 \cdot 4} = 78 \text{ inch units}^3.$$

Now

$$B = \text{max. bending moment} = \frac{WL}{4} = \frac{W \times 96}{4} = 24W \text{ ton ins.}$$

And from $B = f_t Z_t$, see (v), para. 121, we have

$$24W = 1 \cdot 5 \times 78,$$

or

$$W = \frac{1 \cdot 5 \times 78}{24} = 4 \cdot 875 \text{ tons.}$$

We know that

$$\frac{\text{stress on compression edge}}{\text{stress on tension edge}} = \frac{y_c}{y_t} = \frac{7 \cdot 1}{3 \cdot 4}.$$

Hence

$$\text{Stress on compression edge} = \frac{7 \cdot 1}{3 \cdot 4} \times 1 \cdot 5 = 3 \cdot 13 \text{ tons per sq. in.}$$

Problems XIV

1. Find the values of I and Z for a rectangular beam section of breadth 3 ins. and depth 11 ins.

2. What uniformly distributed load would a timber beam of the section indicated in Prob. 1 carry over a span of 15 ft., if the stress in the beam must be limited to 1000 lbs. per sq. in.?

3. Using the dimensions shown in Fig. 184, find the value of the compression modulus (Z_c), given that I_{XX} = 265·6 inch units.

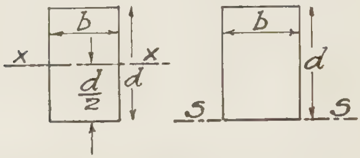
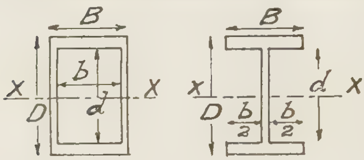
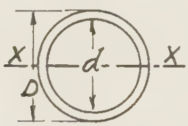
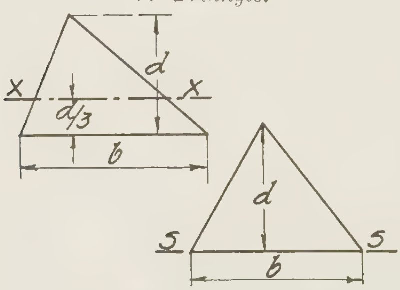
4. Find I_{SS} for the triangle shown in Fig. 185, given that $I_{ZZ} = bd^3/4$.

5. Find I_{XX} for the cast iron beam section shown in Fig. 184. Use the Principle of Parallel Axes.

6. Find the safe load per sq. ft. which could be carried on a timber floor constructed of 9 in. by 3 in. joists, spaced at 15 ins. centres over a span of 12 ft. The stress in the joists to be limited to 900 lbs. per sq. in.

Table V

Moment of Inertia (I), Modulus of Section (Z) and Position of Centroid for various Sections.

Section	Moment of inertia, I	Modulus of section, Z
I. *Rectangle.* 	$I_{XX}=\frac{bd^3}{12}$. $I_{SS}=\frac{bd^3}{3}$.	$\frac{bd^2}{6}$. —
II. *Hollow Rectangle or* **I**-*section* 	$\frac{BD^3-bd^3}{12}$.	$\frac{BD^3-bd^3}{6D}$.
III. *Circle.* 	$\frac{\pi D^4}{64}$.	$\frac{\pi D^3}{32}$.
IV. *Hollow Circle.* 	$\frac{\pi}{64}(D^4-d^4)$.	$\frac{\pi}{32D}(D^4-d^4)$.
V. *Triangle.* 	$I_{XX}=\frac{bd^3}{36}$. $I_{SS}=\frac{bd^3}{12}$.	— —

Note. The axis X–X passes through the centroid in each case, while the axis S–S passes through or is tangential to one side.

CHAPTER XV

SHEAR STRESS AND STRAIN. THE DISTRIBUTION OF SHEAR STRESS IN BEAMS

134. Shear stress and strain. The Modulus of Rigidity. In para. 102 we explained briefly what was meant by shear stress, and showed how the intensity of shear stress might be calculated in a simple case; see Fig. 160. As indicated in that example, shear stress is produced on a plane surface when a force acts across, or is tangential to, that surface.

The kind of distortion or strain which accompanies such a stress may be represented in the following manner: if the dotted rectangle $ABCD$ in Fig. 188 represents a very small block of some material which is subjected to a force S acting across its upper surface AB, then the block will tend to assume the form shown in solid lines, the surface AB moving relatively to DC a distance of, say, x. Then **the shear strain on the block is measured by the angular displacement θ.** If h be the distance between the parallel surfaces AB and CD then, since the angle θ is usually very small, its value will be given in radians by the ratio x/h, or

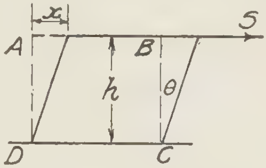


Fig. 188. Shear strain.

$$\textbf{Shear strain} = \theta \text{ (radians)} = x/h. \qquad \text{......(i)}$$

The shear modulus, which we shall denote by the letter G, is then given by the expression:

$$\textbf{Shear modulus} = G = \frac{\textbf{shear stress}}{\textbf{shear strain}}, \qquad \text{......(ii)}$$

the result being given in lbs. per sq. in. or tons per sq. in. according to the units employed.

The constant G, which has a value of about 12,000,000 lbs. per sq. in. for structural steel, is not required in the ordinary range of structural calculations. Its value is usually obtained experimentally from bars subjected to torsion (twisting).

To illustrate the preceding explanation the following experiment may be conducted.

Experiment. *To find* G *for a block of rubber.*

A block of rubber, of which the length is considerable when compared with its depth, is secured by means of a suitable cement to a board as

shown in Fig. 189. To its upper edge a metal bar is similarly attached, and from the end of the bar a cord passes over a pulley; varying weights are attached to the cord. The movement of the upper surface of the block in relation to the lower is measured by a vernier attached to the upper surface, which slides past a fixed scale. To obtain good values, the averages of the readings of the scale are taken as the loads increase and again as the loads decrease. The values of h, x and W are then tabulated. The *shear stress* at any load is obtained by dividing the load W by the area of the upper surface of the block. G is calculated from the average values using

$$G = \frac{\text{stress}}{\text{strain}}.$$

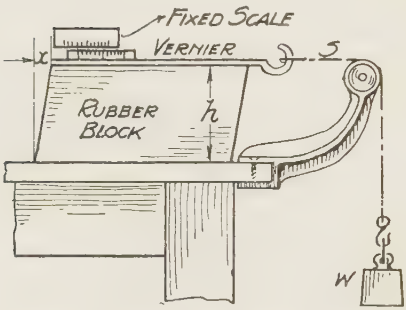


Fig. 189. Finding G for a block of rubber.

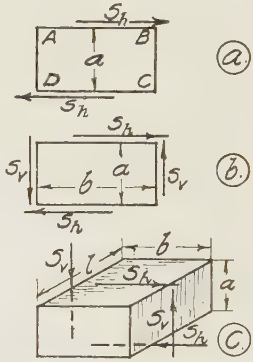


Fig. 190. Complementary shear.

135. Complementary shear stress. If we consider the equilibrium of the small rectangular block $ABCD$ shown in Fig. 190 (a), which is subjected to a shear force S_h acting over its upper surface, we may assume—remembering the small dimensions—that the force S_h acting to the right is balanced by another and equal force S_h acting to the left on the lower surface CD of the block. But, if the altitude of the block be a, these two forces will constitute a couple, of which the moment will be equal to $S_h \times a$, and this will be insufficient to maintain the block in equilibrium. Now we know that the couple just defined can only be balanced by another couple, of equal value but opposite effect, so that there must exist equal tangential forces, of magnitude S_v, acting over the vertical surfaces BC and DA, see Fig. 190 (b). If b be the width of the block, then for equilibrium the couple formed by the latter two forces must have a value equal to the original couple, that is

$$S_v \times b = S_h \times a.$$

Now we have spoken of $ABCD$ as a very small block of material, and if l represents its length, see Fig. 190 (c), and s be the shear

stress acting over the upper and lower surfaces, while s_1 be that acting over the two vertical surfaces, we have that

$$S_h = \text{total horizontal shear force}$$
$$= \text{area} \times \text{stress} = bl \times s.$$

Similarly
$$S_v = \text{total vertical shear force}$$
$$= al \times s_1.$$

But, as we have seen, the couples which these forces form must be equal, so that $S_h \times a = S_v \times b$, and, substituting the values of S_h and S_v obtained above, we have

$$(s \times bl)\, a = (s_1 \times al)\, b,$$

whence $\quad s\,(bla) = s_1\,(bla),$

so that $\quad s = s_1,$

that is, the intensity of shear stress on the surfaces at right angles to each other are equal. Such a state of stress is known as Simple Shear, so that:

I. In a state of simple shear, a shear stress acting in one plane is accompanied by a complementary shear stress of equal intensity in a plane at right angles to the first plane.

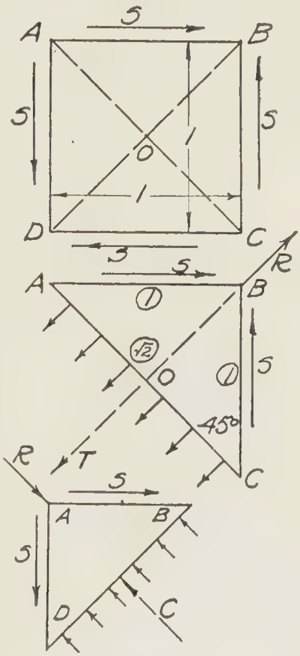


Fig. 191.

Next let us consider a small *cube* of material, such as *ABCD* in Fig. 191, which is in a state of simple shear, the shear intensity over each surface being equal to s. Then if the dimensions of each surface of the block be equal to unity, the total shear force acting over each surface will be equal to $s \times 1 \times 1 = s$. Now consider the equilibrium of the portion *ABC*, which is obtained by imagining the block divided along one of the diagonals *AC*. Then the two shear forces s and s acting towards B may be resolved into their rectangular components acting in the directions of the diagonals *OB* and *AC*.

Now since $\quad$ angle BCA = angle $BAC = 45°$,

therefore
$$\frac{BC}{AC} = \frac{AB}{AC} = \cos 45° = \frac{1}{\sqrt{2}} = \frac{OB}{BC}.$$

Then the force R, which is equal to twice the component of s along OB,

$$= 2 . s \left(\frac{OB}{BC}\right) = 2 . s \left(\frac{1}{\sqrt{2}}\right) = \frac{2s}{\sqrt{2}}, \text{ or } R = \sqrt{2} . s. \quad \ldots\ldots(\text{a})$$

This force must evidently be balanced by an equal force, say T, acting in the opposite direction and which is the result of a tensile stress, say t, acting over the surface AC, or $T = t \times$ (area of AC). Also the area of the surface AC will equal

$$\frac{\text{area } BC}{\cos 45^\circ} = \frac{1}{\frac{1}{\sqrt{2}}} = \sqrt{2}.$$

Then $$\text{force } T = \sqrt{2} . t. \quad \ldots\ldots(\text{b})$$

Then since, for equilibrium, the forces R and T must be equal,

or $$R = T,$$

we have, from (a) and (b) above,

$$\sqrt{2} . s = \sqrt{2} . t,$$

so that evidently $$s = t,$$

or the intensity (t) of the tensile stress across AC is equal to the intensity (s) of the original shear stress. By a similar process of reasoning we may show that the intensity of stress across the other diagonal BD is also equal to s, but is compressive in this case; see Fig. 191. Finally, by resolving the forces s *along* the diagonal surfaces AC and BD, we may show that in each case they balance each other, and that the shear stresses on these surfaces are zero.

II. **Hence we conclude that, in a state of simple shear of intensity s, tensile and compressive stresses of the same intensity are induced on planes at 45° to those of pure shear.**

III. The converse statement, which is also true, is sometimes useful; we may thus say that, **if a material be subjected to compressive and tensile stresses of equal intensity acting on planes at right angles to each other, then pure shear stresses of the same intensity will be induced on planes at 45° to the original planes.**

A knowledge of these relations is important when we come to consider the failure of certain materials subjected to compressive stress, and also when dealing with the strength of reinforced concrete.

136. Shear stress in beams. We have seen (Chap. VIII) that in a loaded beam or cantilever the action of the load, see Fig. 128, produces at any vertical section first a couple (the moment of

resistance, which is made up of two equal and opposite horizontal forces T and C), and second an internal vertical shear force S, which is evidently the result of vertical shear stresses acting over the section of the beam, and resisting the tendency of the external loading to shear one portion of the beam from the other. In addition, we know that *such vertical shear stresses must be accompanied by horizontal shear stresses of equal intensity at each horizontal section* (see I above).

It may be imagined that, in order to obtain the intensity of the shear stress across any vertical section of a loaded beam, we need only divide the total shear force by the area of the section; this, however, is not the case. The distribution of the shear stress is not uniform and, as will presently be seen, a somewhat lengthy investigation is necessary before we can say how the shear stresses are distributed over the section.

Mean shear stress. The value obtained by dividing the total shear force (S) by the area (A) of the section has, however, some practical value; it is usually called the **Mean Shear Stress** and we may denote it by s_m, so that

$$s_m = \text{mean shear stress} = \frac{S}{A}. \qquad \text{......(iii)}$$

Shear stress at the neutral axis of a beam. Let AB and CD in Fig. 192 be two vertical sections of a loaded beam, the sections being a very short distance x apart, and the cross section of the beam being uniform between the two sections and symmetrical about the vertical axis YY. If the bending moments at AB and CD be B_1 and B_2 respectively, then the change between the bending moment at AB and that at CD is $B_1 - B_2$. Then, since the distance x is very small, the *rate of change of bending moment* between these two sections will be given by $(B_1 - B_2)/x$, and this is equal to the shear force (S) over this portion (see II, para. 92), or

$$S = \frac{B_1 - B_2}{x}, \qquad \text{......(a)}$$

If f_1 and f_2 be the maximum compressive stresses induced at sections AB and CD respectively, we may set out two stress diagrams as shown in Fig. 192, the magnitudes of f_1 and f_2 being obtained from the expression $f = By/I$, where y is the distance from X–X to the compression edge, and I is the moment of inertia of the section.

Let A be the area of the section at AB above the neutral axis X–X. Since the section of the beam does not change, A is also the area above X–X at section CD. Let the total forces acting on these areas be C_1 on area NA and C_2 on area at CL, as indicated in the stress diagrams.

Now consider the horizontal equilibrium of the portion of the beam $ANLC$. Since C_1 and C_2 are not equal (otherwise S, the shear force, would be zero), there must, for equilibrium, be a horizontal force acting over the surface at NL, the neutral layer, equal to the difference between C_1 and C_2. Let this force be C_n. Then *the force* C_n *is evidently supplied by the horizontal shear stresses acting over the area at* NL; this area being equal to xb_n, where b_n is the width of the section at X–X, the neutral axis; see Fig. 192.

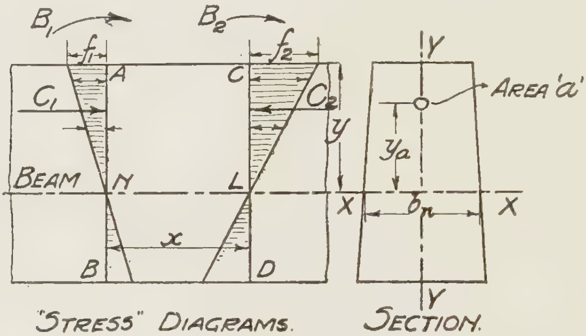


Fig. 192. Shear stress at the neutral axis of a beam.

Then, if s_n be the intensity of shear stress at the neutral axis, we have

$$C_n = C_1 - C_2 = s_n\,(xb_n). \qquad \text{......(b)}$$

To find the magnitude of the forces C_1 and C_2, consider a small element of area a, situated on the section AB at a distance y_a from the neutral axis. The intensity of stress on this area will be given by

$$f_a = \frac{B_1 y_a}{I},$$

and the total force on the small area

$$= a \times f_a = a\left(\frac{B_1 y_a}{I}\right).$$

So that the *total force* C_1, acting on the area above the neutral axis at section AB, will be given by

$$C_1 = \Sigma\, a\left(\frac{B_1 y_a}{I}\right) = \frac{B_1 YA}{I},$$

where Y is the distance to the centroid of the area A from the neutral axis X–X.

Similarly $\qquad$ the force $C_2 = \dfrac{B_2 YA}{I}$.

Then, substituting these values in (b) above, we have

$$C_n = s_n\,(xb_n) = C_1 - C_2 = \frac{B_1YA}{I} - \frac{B_2YA}{I},$$

$$\therefore\ s_n\,.\,x\,.\,b_n = \frac{YA}{I}\,(B_1 - B_2),$$

from which we have

$$s_n = \frac{YA}{Ib_n}\left(\frac{B_1 - B_2}{x}\right).$$

But from (a) we have that $(B_1 - B_2)/x$ is equal to S, the shear force over the section; hence we have

$$\textbf{Shear stress at neutral axis} = s_n = \frac{SYA}{Ib_n}, \quad \text{......(iv)}$$

where S is the total shear force acting at the section, A is the area of the section above the neutral axis, Y is the distance from the neutral axis to the centroid of the area, I is the moment of inertia of the section, and b_n is the width of the section at the neutral axis. This is a general expression and may be applied to sections of any shape.

Example. *To find an expression for shear stress at the neutral axis of a rectangular beam section.* See Fig. 193.

In this case A = area above neutral axis $= b \times d/2$; Y = distance to the centroid of this area $= d/4$, see Fig. 193; $b_n = b$, and $I = bd^3/12$. Then s_n, the shear stress at the neutral axis,

$$= \frac{SYA}{Ib_n} = \frac{S \times d/4 \times bd/2}{b \times bd^3/12} = \frac{3S}{2bd},$$

but the mean shear stress on this section (see (iii) above) $= s_m = S/bd$; hence we have that

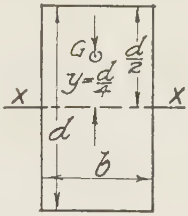


Fig. 193.

The shear stress at the neutral axis of a rectangular beam

$$= \frac{3}{2}\frac{S}{bd} = \frac{3}{2}s_m, \quad \text{......(v)}$$

where s_m is the mean shear stress. (See also V, para. 137.)

137. Distribution of shear stress in beams. Taking, as before, two sections in a beam, AB and CD, a short distance x apart, for which the stress-distribution diagrams are as shown in Fig. 194 (A), consider the equilibrium of the portion of the beam $AEFC$ above the layer EF, where EF is at a distance y_e from NL the neutral layer.

If C_1' and C_2' represent the total forces acting on the vertical surfaces of this portion, to the left and right respectively, then the magnitude of these forces will evidently be given by the "volume"

of the stress-distribution diagrams above the plane of EF; see para. 125. Also, for equilibrium, since C_1' and C_2' are not equal, there must be a horizontal shear force (say C_d') acting along the surface at EF, which is equal to the difference between the two forces C_1' and C_2', or

$$\text{Total shear force on plane } EF = C_d' = (C_1' - C_2'). \quad \ldots\ldots(a)$$

The area of the surface at EF will equal $(b_e x)$, where b_e is the width of the section at the level of EF.

Then the intensity of shear stress at EF

$$= \frac{\text{shear force}}{\text{area}} = \frac{C_d'}{b_e x}. \qquad \ldots\ldots(b)$$

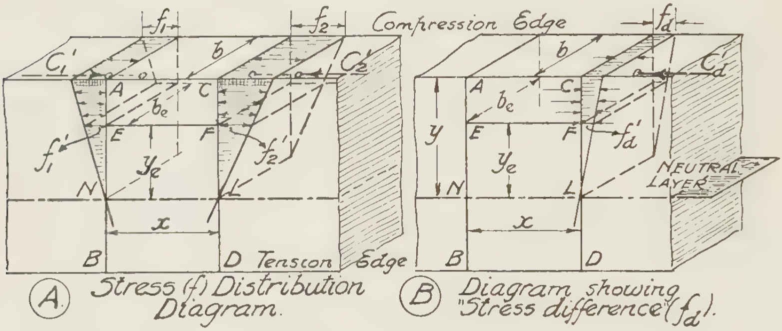


Fig. 194. Distribution of shear stress in beams.

To find the magnitude of C_d' let a new stress-distribution diagram be constructed as at (B) Fig. 194, in which the maximum stress (f_d) is equal to the difference between the stress f_1 and the stress f_2, or $f_d = (f_1 - f_2)$.

(**Note.** Since the stresses f_1 and f_2 were obtained from B_1 and B_2—the bending moments at sections AB and CD respectively—using the expression $f = By/I$, then f_d may be obtained conveniently from the expression

$$f_d = \frac{B_d y}{I}, \qquad \ldots\ldots(c)$$

where $B_d = (B_1 - B_2)$ = the change in the bending moment over the distance x. Also, from this, $B_d = f_d Z$.)

This new diagram may be called the **Diagram showing distribution of stress-difference** (f_d); thus at any horizontal layer such as EF, the stress-difference (f_d') represented on this diagram must, by the construction of the diagram, be equal to the difference

between the corresponding stresses (f_1' and f_2') on the original stress diagrams at (A) Fig. 194, or

$$f_d' = (f_1' - f_2').$$

From this it follows that the total force, represented by the "volume" of the stress-difference diagram above the plane of EF, must be equal to the difference between the forces C_1' and C_2', and represented by the corresponding portions of the stress distribution diagrams at (A). As indicated in (a) above this force is evidently equal to the shear force C_d' acting over the surface at EF, and we may conclude that

IV. If a diagram showing the distribution of stress-difference be drawn as explained above, then the total shear force acting on any horizontal layer in the beam, between the two selected vertical sections, will be represented by the "volume" of the stress-difference diagram above that layer.

This is a general statement and may be applied to sections of any shape. Having obtained the total horizontal shear force on the plane EF in this way, we may find the intensity of shear stress on this plane from

$$\text{Shear stress} = \frac{\text{horizontal shear force}}{\text{area}} = \frac{C_d'}{b_e x}. \quad \text{......(d)}$$

It will usually be convenient to make x equal to unity, either 1 ft. or 1 in. The following Example will explain the method of procedure.

Example. *Find the distribution of shear stress at any section* F *between points* A *and* C *on the loaded beam shown in Fig.* 195 (A), *if the section of the beam is a rectangle* 6 *ins. broad* (b) *and* 12 *ins. deep* (d).

In this case the shear force S is constant between A and C, so that it is immaterial where the two sections are taken.

The selected sections A and F are 1 ft. apart. Then

B_a = bending moment at $A = 0$

and B_f = bending moment at $F = 1500 \times 12 = 18{,}000$ lb. ins.

Then B_d = difference in bending moment over 1 ft.

= rate of change of bending moment over 1 ft., or 12 ins.

= 18,000 lb. ins.

For this section the value of Z

$$= \frac{bd^2}{6} = \frac{6 \times 12^2}{6} = 144 \text{ inch units}^3.$$

Then from $B_d = f_d Z$, we have

$$f_d = \text{maximum "stress-difference"} = \frac{B_d}{Z} = \frac{18{,}000}{144}$$

$$= 125 \text{ lbs. per sq. in.}$$

The diagram showing the distribution of stress-difference may now be set out as shown in Fig. 195 (B). Since the section is symmetrical we need only consider the upper half. On this we may take three parallel planes at EE, GG and XX, 2 ins. apart in each case.

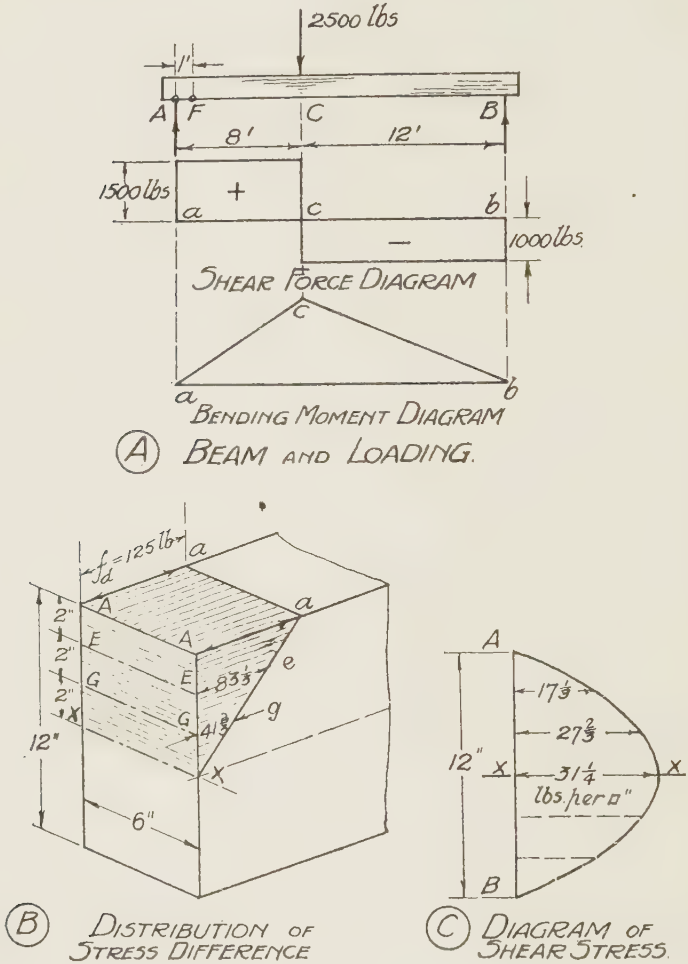


Fig. 195. Distribution of shear stress on a rectangular section.

On this diagram the length Aa represents f_d, that is 125 lbs. per sq. in. Hence on plane EE the length Ee will represent $(125 \times \frac{4}{6})$ or $83\frac{1}{3}$ lbs. per sq. in., and on plane GG the length Gg represents $(125 \times \frac{2}{6})$ or $41\frac{2}{3}$ lbs. per sq. in. At XX the stress-difference is zero.

Then the total horizontal shear force acting on plane EE is represented by the "volume" of the stress-difference diagram above this plane, that is by the "solid" $EEAAaa$.

Hence the total force acting on plane EE

$$= \left(\frac{125 + 83\frac{1}{3}}{2}\right)(6 \times 2) = (208\tfrac{1}{3} \times 6),$$

and this acts over an area of $(b_e x)$, i.e. (6×12) or 72 sq. ins.

Then shear stress on plane $EE = 208\frac{1}{3} \times 6/72 = 17\frac{1}{3}$ lbs. per sq. in.

Similarly on plane GG the shear stress

$$= \left(\frac{125 + 41\frac{2}{3}}{2}\right)(6 \times 4)\frac{1}{72} = 27\tfrac{2}{3} \text{ lbs. per sq. in}$$

And the shear stress on plane XX

$$= \frac{125}{2}(6 \times 6)\frac{1}{72} = 31\tfrac{1}{4} \text{ lbs. per sq. in.}$$

Let these values be now plotted on a Shear Stress Diagram as shown in Fig. 195 (C), on a base line A -B representing the depth of the beam, the same values being used below the neutral axis at X; then the points will be found to lie on a smooth curve and the maximum value will occur at X. This curve may be readily shown to be a part of a parabola.

Checking by the expression (iii), found above in para. 136, we have

$$s_n = \text{shear stress at neutral axis} = \frac{3}{2}\frac{S}{bd}$$

$$= \frac{3}{2} \times \frac{1500}{6 \times 12} = 31\tfrac{1}{4} \text{ lbs. per sq. in.}$$

The following general statement may be based upon the results of the above Example:

V. In a rectangular beam section the distribution of shear stress is given by a parabolic curve, having zero values at the top and bottom edges respectively, and a maximum value at the neutral axis of one and a half (3/2) times the mean shear stress (S/A) over the whole section.

Other sections. In the case of a **circular section** it may be shown that the shear stress varies in a similar way, *the maximum value at the neutral axis being one and one-third* (4/3) *times the intensity of the mean shear stress* (S/A). In the case of a **thin circular pipe** *the maximum shear stress at the centre is twice the mean shear stress.*

138. Distribution of shear stress in I-section beams. Using statement IV above we may construct a shear stress diagram for **I**-section beams, the procedure being explained by the following worked Example.

Example. *Draw a diagram showing the maximum shear stresses in a beam of the section shown in Fig.* 196 (B), *the beam being loaded with a uniformly distributed load of* 2 *tons as shown in Fig.* 196 (A).

In this case the maximum shear forces occur at the points of support, and fall off uniformly to zero at the centre. The maximum shear stresses

will therefore occur in the sections immediately over the points of support. At these sections the shear force is 2240 lbs. Evidently this force also gives *the rate of change of the bending moment* (*per unit of length*) at the same section (see II, para. 92), which must likewise be a maximum at this section; or B_d = maximum change in bending moment = 2240 lb. in. (per inch of length).

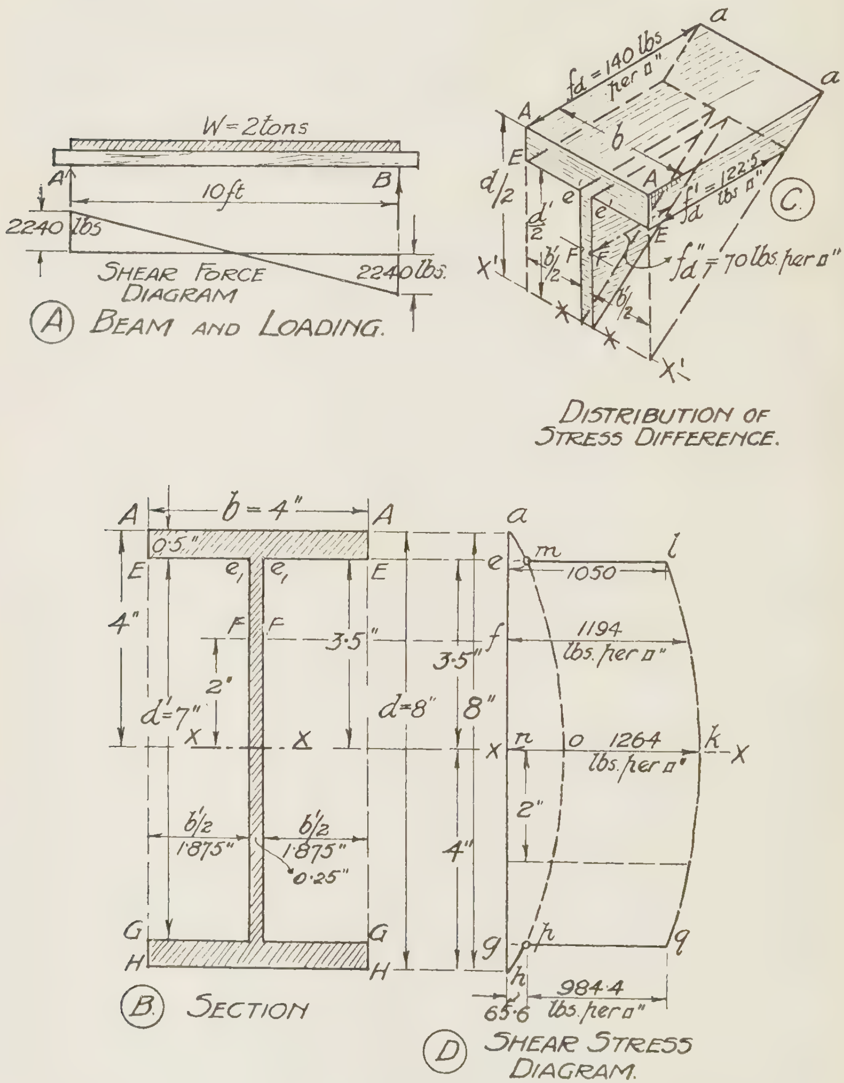


Fig. 196. Distribution of shear stress in an I-section beam.

The moment of inertia I for the section, by the method explained in para. 129,

$$= \frac{4 \times 8^3}{12} - \frac{3 \cdot 75 \times 7^3}{12} = 64 \text{ inch units}^4 \text{ (approx.)}.$$

Then $$f_d = \text{max. stress-difference} = \frac{B_d y}{I}$$

$$= \frac{2240 \times 4}{64} = 140 \text{ lbs. per sq. in.}$$

The diagram of distribution of stress-difference may now be constructed as shown in Fig. 196 (C), only one-half being necessary. Its shape may be looked upon as made up of a wedge-shaped solid $aaAAX'X'$, from which is taken the smaller wedge-shaped solids, indicated by dotted lines; the volume of the solid thus formed may be most readily calculated by dealing with it in the following order.

The length Aa represents a stress-difference (f_d) of 140 lbs. per sq. in., so that the stress-difference (f_d') at EE will be ($140 \times 3 \cdot 5/4$) or 122·5 lbs. per sq. in. Taking a third plane at FF, 2 ins. from XX, the stress-difference (f_d'') at FF will be ($140 \times \frac{2}{4}$) or 70 lbs. per sq. in.

(*a*) The total shear force acting on the plane EE is represented by the "volume" of the stress-difference diagram above this plane, or, otherwise, is equal to (the area of the flange) × (the average stress acting thereon), the average stress being given by

$$\left(\frac{f_d + f_d'}{2}\right).$$

Therefore

$$\text{Total force on } EE = \left(\frac{140 + 122 \cdot 5}{2}\right)(4 \times 0 \cdot 5) = 262 \cdot 5 \text{ lbs.}$$

If we consider the shear stress on the plane *just above* EE, the area of which is (4×1) or 4 sq. ins. (since the distance x in this case is 1 in.), then

$$\text{Shear stress just above } EE = \frac{262 \cdot 5}{4} = 65 \cdot 6 \text{ lbs. per sq. in.}$$

(*b*) Similarly the shear stress *just below* EE, where the width $e'e'$ is only 0·25 in. and the area ($1 \times 0 \cdot 25$), is given by

$$\text{Shear stress just below } EE = \frac{262 \cdot 5}{0 \cdot 25} = 1050 \text{ lbs. per sq. in.}$$

(*c*) The total shear force acting on plane FF will be given by the total force acting on the flange *plus* the force represented by "volume" of the stress-difference figure between the flange and the plane FF. Then

$$\text{Total force on plane } FF = 262 \cdot 5 + \left(\frac{122 \cdot 5 + 70}{2}\right)(1 \cdot 5 \times 0 \cdot 25)$$

$$= 262 \cdot 5 + 36 = 298 \cdot 5 \text{ lbs.}$$

This is spread over an area of ($1 \times 0 \cdot 25$) or 0·25 sq. in. Therefore

$$\text{Shear stress at } FF = \frac{298 \cdot 5}{0 \cdot 25} = 1194 \text{ lbs. per sq. in.}$$

(*d*) The total horizontal force acting on the plane XX will be given

by the total "volume" of the stress-difference diagram above XX, which, using the letters and dimensions given in Fig. 196 (B) and (C),

$$= \frac{1}{2}\left(b \times \frac{d}{2} \times f_d\right) - \frac{1}{2}\left(b' \times \frac{d'}{2} \times f_d'\right)$$
$$= \left(\frac{4 \times 4 \times 140}{2}\right) - \left(\frac{3{\cdot}75 \times 3{\cdot}5 \times 122{\cdot}5}{2}\right)$$
$$= 1120 - 804 = 316 \text{ lbs.}$$

This acts over an area of $(1 \times 0{\cdot}25)$ or $0{\cdot}25$ sq. in.; hence

$$\text{Shear stress at } XX \text{ (the neutral axis)} = \frac{316}{0{\cdot}25} = 1264 \text{ lbs. per sq. in.}$$

The shear stress diagram can now be constructed as shown in Fig. 196 (D). It may be shown that the curves *amoph* and *lkq* are identical curves, both being parts of the same parabolic curve; this fact may sometimes be used in setting out the curves.

(It may be noted here that the reason for bevelling the flanges, and rounding the internal angles of beam and other steel sections, is not merely to reduce the difficulties of rolling—by providing "clearance" for the mill rollers—but also to avoid sudden and therefore undesirable changes in the shear stresses, which would otherwise occur at each sudden change of section; e.g. in Fig. 196, if the section actually had sharp angles as therein shown, there would be a sudden alteration of stress at the plane EE from 65 to 1050 lbs. per sq. in. The rounding of the internal corners ensures a more gradual change.)

139. Approximate expression for the mean shear stress in the web of an I-section or built-up beam. The total shear force which is borne by the web in the above Example will be given by the product of the area of the figure *elqg* into the width of the web. This product will be seen to be a very high proportion of the total shear force (S) carried by the whole section, the stresses in the flanges—found from the small parts of the shear stress diagram beyond *em* and *gp*—being very small indeed. On this fact is based the following approximate rule, which is frequently used in practice and which assumes that the whole of the shear force (S) is carried by the web.

VI. The approximate mean shear stress in the web of an I-section beam may be found by dividing the total shear force by the area of the web, or

$$\text{approximate mean shear stress} = \frac{S}{\text{area of web}}.$$

By way of comparison we may find the approximate mean shear stress in the example above.

$$\text{Approx. mean shear stress in web} = \frac{\text{shear force}}{\text{area of web}}$$
$$= \frac{2240}{7 \times 0{\cdot}25} = 1280 \text{ lbs. per sq. in.}$$

(Result in para. 138 was 1264 lbs.; difference equals 1·3 %.)

140. To find the distribution of shear stress in beams by means of the Calculus*; see Fig. 197.

Let two sections AB and CD be taken as before, a small distance δx apart, and let the bending moment at AB be B, while that at CD is equal to $(B + \delta B)$. Then the difference in bending moment equals δB over a distance δx; hence

$$S = \text{shear force} = \text{rate of change of bending moment} = \frac{\delta B}{\delta x}. \quad \text{......(a)}$$

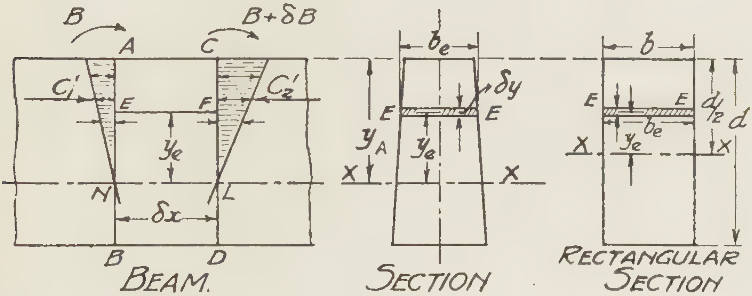


Fig. 197.

Consider a very thin strip at EE on section AB, of a thickness δy and at a height y_e above the neutral axis X–X, the breadth of the figure being b_e at this height.

The stress acting on this strip

$$= f_e = \frac{By_e}{I}. \quad \text{......(b)}$$

The total force acting on the strip

$$= \left(\frac{By_e}{I}\right)(b_e \delta y). \quad \text{......(c)}$$

Similarly the total force acting on the corresponding strip at F

$$= \left(\frac{B + \delta B}{I}\right) y_e \, (b_e \delta y). \quad \text{......(d)}$$

Now the shear force acting on the horizontal surface at EF, and due to the forces acting on the elemental strips at E and F, must be equal to the difference between the forces defined in (c) and (d), and this difference

$$= \frac{\delta B \,.\, y_e \,.\, b_e \,.\, \delta y}{I}. \quad \text{......(e)}$$

The *total forces* C_1' and C_2', see Fig. 197, acting on the vertical surfaces AE and CF above the layer EF, may be obtained by summing the forces acting on the elemental strips, such as EE, over each of these areas. Similarly the *total difference* between these two forces may be obtained by summing the quantities defined above in (e) over the same area

* This paragraph may be omitted by readers not acquainted with the use of the Calculus, or for whom the methods already outlined are sufficient.

above EE (or FF). Then, if y_A is the distance from the neutral axis $X-X$ to the compression edge, we have

Total shear force acting on the horizontal surface EF

$$= \int_{y=y_e}^{y=y_A} \frac{\delta B \,.\, y \,.\, b_e \,.\, \delta y}{I} = \frac{\delta B}{I} \int_{y=y_e}^{y=y_A} y \,.\, b_e \,.\, \delta y. \qquad \text{......(f)}$$

Since the force defined in (f) acts over an area of $(b_e \delta x)$, then

$$\text{Shear stress on surface at } EF = \frac{\text{shear force}}{\text{area}} = \frac{\delta B}{I\,(\delta x \,.\, b_e)} \int_{y=y_e}^{y=y_A} y \,.\, b_e \,.\, \delta y;$$

and, since $\delta B/\delta x = S$, see (a) above, then

$$\textbf{Shear stress on } \boldsymbol{EF} = \frac{S}{Ib_e} \int_{y\ =}^{y=y_A} y \,.\, b_e \,.\, \delta y. \qquad \text{......(vi)}$$

This is a general expression and may be applied to sections of any shape. By means of it we are able to find the shear stress on any horizontal surface such as EF at a distance of y_e from the neutral axis.

It should be noted that the second portion of the expression (vi), viz.

$$\int_{y=y_e}^{y=y_A} y \,.\, b_e \,.\, \delta y,$$

should give the first moment of the area of the section above EE (or FF) about XX. Further, if the section of the beam does not vary *uniformly* in width from either edge to the neutral axis, but is subject to sudden changes, as in an **I**-section, then this portion of the expression should be split up into two (or more) parts, appropriate values being given to y_A, y_e and b_e, in each case.

The value of b_e outside the integral sign (S/Ib_e) should be equal to the width of the section *at the plane upon which the shear stress is being obtained.*

Example. *Find the distribution of shear stress on a rectangular section by the above methods.* See Fig. 197.

In this case

$$b_e = b, \quad y_A = \frac{d}{2}, \quad \text{and} \quad I = \frac{bd^3}{12}.$$

Then expression (vi) becomes

$$\text{Shear stress on layer at } EE = \frac{S}{\dfrac{bd^3}{12} \times b} \int_{y=y_e}^{y=\frac{d}{2}} y \,.\, b \,.\, \delta y.$$

From this it is easy to obtain the distribution of the stress over the whole section. To find the maximum value at XX the lower limit becomes 0, and we have

$$\text{Shear stress at neutral axis} = \frac{S}{\dfrac{b^2d^3}{12}} \int_{y=0}^{y=\frac{d}{2}} y \,.\, b \,.\, \delta y$$

$$= \frac{S}{\dfrac{b^2d^3}{12}} \left[\frac{y^2}{2} \,.\, b\right]_0^{\frac{d}{2}} = \frac{S \times 12}{b^2d^3} \cdot \frac{d^2}{8} \,.\, b$$

$$= \frac{3}{2} \cdot \frac{S}{bd}, \text{ as in (v) above.}$$

141. Deflection of a beam due to shear. If in Fig. 188 the block $ABCD$ be considered to form part of a cantilever, projecting outwards from its support at DC, it will be seen that the effect of the shear force S is to cause the outer end AB to drop below the level of the inner end at DC. This movement is actually very slight, but it will be readily conceived that, if this effect is continued throughout the length of a cantilever, the drop or deflection at the end of the cantilever may be quite appreciable. Unless, however, the cantilever is both short and deep, this movement may be shown to be relatively small when compared with the deflection due to direct bending. In all ordinary cases the deflection due to shear both in cantilevers and beams may be ignored.

Problems XV

1. In the experiment described in para. 134 and illustrated in Fig. 189, the dimensions of the block of rubber were: length 12 ins., height (h) 3 ins. and thickness $1\frac{1}{2}$ ins. For a load (W) of 20 lbs. the deformation (x) was 0·02 in. Calculate the value of G, the shear modulus of elasticity for the rubber, from these values.

2. A timber beam of uniform rectangular section, 4 ins. broad (b) and 8 ins. deep (d), carries a central load of 2 tons. Find the maximum shear stress induced in the beam. What is the value of the mean shear stress at the same section?

3. If the longitudinal shear stress in a timber beam, 10 ins. by 4 ins. in section, is limited to 100 lbs. per sq. in., find the total shear force which the beam will carry.

4. Find the greatest span over which the beam in Prob. 3 would carry a load equal to $2S$, where S is the maximum shear force which the section will carry, if the flexural stress (f) in the beam is limited to 1100 lbs. per sq. in.

5. It is desired to design a timber beam for an experiment on longitudinal shear, the beam to fail if possible by longitudinal shear near the ends. If the section of the beam be 2 ins. (b) by 6 ins. (d), calculate a suitable span and central load so that, when the flexural stress reaches 750 lbs. per sq. in., the maximum shear stress shall be 120 lbs. per sq. in.

6. Using the approximate rule (para. 139), find the maximum shear force (S) which the **I**-section in Fig. 196 will carry, if the shear stress in the web is to be limited to $2\frac{1}{2}$ tons per sq. in.

7. Draw the diagram of shear stress for the beam illustrated in Fig. 195, if it is to carry a central load of 6000 lbs. What would be the maximum shear stress at this load?

8. Using the method described in para. 138, draw the complete shear stress diagram for the **I**-section shown in Fig. 183, if the maximum shear force (S) is 6 tons. What is the maximum shear stress?

9. Solve Prob. 8 by the calculus method described in para. 140.

CHAPTER XVI

THE DEFLECTION OF BEAMS

142. Deflection and stiffness. As a matter of common experience we know that a beam bends on being loaded. In para. 117 we obtained an expression (i) in which was included a term R, representing the radius of curvature to which the loaded beam had been bent over the very short length there considered. In practice, however, it is not so much the actual shape of the bent beam which we require to know, as the magnitude of the movement of each section of the beam away from its original unstrained position. In particular we wish to know the maximum movement which takes place in this way.

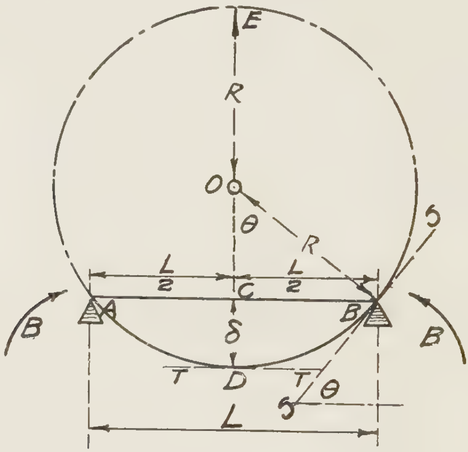


Fig. 198. Beam bending to an arc of a circle.

If in Fig. 198 the line ACB represents a beam which is to be symmetrically loaded about its centre point C, while the line ADB represents the shape assumed by the beam after loading (these two lines may be taken to coincide with the neutral layer of the beam), then CD, which is the amount by which the centre section has moved from its original position, is obviously the maximum movement or deflection of any section of the beam. The term **Deflection** is usually used to denote this maximum deflection and is represented by the Greek letter δ (*delta*).

In order to keep our investigations within the limits of difficulty which we have set for this volume, we will confine our attention to

the maximum deflections of cantilevers and of beams symmetrically loaded. While the methods outlined below for dealing with such cases should prove to be reasonably simple, they are fundamentally sound, so that, should the reader desire to do so, it will not be difficult to extend them to cover more complex cases than are dealt with in this volume.

It may be noted here that, in a beam which is not symmetrically loaded, the maximum deflection does not occur at the centre. In such cases the amount of the maximum deflection will be given by the distance between the line AB, see Fig. 198, and some line TT drawn parallel to AB which touches or is tangential to the curve of the beam. The point of contact of this line with the curve is of course the point at which the maximum deflection occurs.

143. Deflection of a beam bending to circular arc. In Fig. 198 is shown a beam ACB which, when subjected to a constant bending moment (B) throughout its length, bends to the curve ADB; see example given in para. 115. Then, from the expression (iv) given in para. 120, we have

$$\frac{B}{I} = \frac{E}{R}.$$

Now in the present case, if the beam be of uniform section throughout its length, then B, I and E are all of constant value, and it follows from the above expression that the term R must be constant over the length of the beam. *If a curve has a radius of curvature* (R) *which is constant, then the curve must be part of a circle.* This circle has been completed in Fig. 198 and from it, using a well-known geometrical principle, we may show that

$$AC \times CB = CE \times CD, \qquad \text{......(a)}$$

but

$$AC = CB = L/2.$$

Also, since the distance CD (that is δ) is usually very small compared with the distance CE, we may assume that $CE = DE = 2R$. Then, substituting these values in (a) above, we have

$$\left(\frac{L}{2}\right)^2 = 2R \times \delta,$$

or

$$\delta = \frac{L^2}{4} \times \frac{1}{2R} = \frac{L^2}{8}\left(\frac{1}{R}\right). \qquad \text{......(b)}$$

But from

$$\frac{B}{I} = \frac{E}{R},$$

we have

$$\frac{1}{R} = \frac{B}{EI}.$$

Hence by substituting this value in expression (b) we have

$$\delta = \frac{BL^2}{8EI}. \qquad \ldots\ldots\text{(i)}$$

Experiment. *Deflection of a beam bending to a circular arc.*

Let a beam of uniform section, for the material of which the value of E is known, be supported at two points A and B as shown in Fig. 199, and loaded with equal loads W at D and C, where the distances CA and BD are both equal to L_1; then, between the points A and B, the

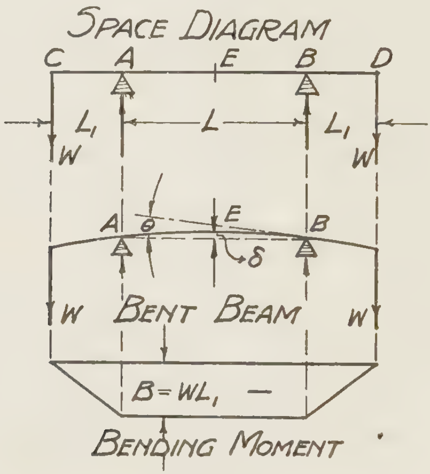


Fig. 199.

beam will be subjected to a constant bending moment (B) equal to $W \times L_1$; see bending moment diagram. If the deflection at E, *which is upwards in this case since the bending moment is negative*, is measured for a number of differing loads W, then the measured deflections may be compared with the calculated deflections using expression (i). (Alternatively the experiment may be used to find the value of E.)

Experiment. *To find* E *for a timber beam of red deal.* See reference in para. 225.

The experiment consists of submitting the timber beam to a constant bending moment, as explained above; from the measured deflection the value of E may be found by using expression (i) above. The test may be carried out on the form of apparatus ordinarily used for measuring the deflection of beams, the deflection being measured upwards at the centre.

Using Fig. 199, the dimensions in a particular test were as follows: L = span = 48 ins.; L_1 = projection of beam at each end = 18 ins.; W = the load hung from each end = 200 lbs. Dimensions of the beam:

breadth (b) = 2 ins., depth (d) = 3 ins. For the end loads of 200 lbs. the measured deflection (δ) at point E at the centre was 0·19 in. Then

$$B = \text{constant bending moment between } A \text{ and } B$$
$$= 200 \times 18 = 3600 \text{ lb. ins.}$$
$$\text{The moment of inertia } (I) = \frac{bd^3}{12} = \frac{2 \times 27}{12} = \frac{27}{6}.$$

Then from expression (i) we have

$$\delta = \frac{B.L^2}{8.E.I},$$

whence E = modulus of elasticity $= \dfrac{B.L^2}{8.I.\delta}$

$$= \frac{3600 \times 48 \times 48 \times 6}{8 \times 0{\cdot}19 \times 27} = 1{,}212{,}000 \text{ lbs. per sq. in.}$$

144. Slope of a beam. If at any point, such as B in Fig. 198, a line SS is drawn which is tangential to the curve of the beam at that point, then that line will evidently give the **Inclination** or **Slope** of the axis of the beam at that point.

In the case illustrated in Fig. 198 we have, if θ be the angle which the line SS forms with a horizontal line (or, more correctly, with the original position ACB of the beam), then, by the geometry of the figure, the angle DOB is also equal to θ. Now, *since θ will usually be very small*, we may say that $\sin\theta = \theta$ (radians), so that

$$\theta \text{ (radians)} = \sin \angle DOB = \frac{BC}{BO}$$
$$= \frac{\frac{L}{2}}{R} = \frac{L}{2}\left(\frac{1}{R}\right). \qquad \text{......(a)}$$

But $\dfrac{1}{R} = \dfrac{B}{EI}$ (see para. 143),

and substituting this value of $1/R$ in (a), we have

$$\theta \text{ (radians)} = \text{slope in radians} = \frac{BL}{2EI}, \qquad \text{......(ii)}$$

which gives the slope at the point B (or A) in a beam subjected to a constant bending moment, or which is bent to a circular arc. (L must of course be reduced if B is inside the point of support.)

Experiment. *To find the change in slope of a beam subjected to a constant bending moment.* See Fig. 200.

In the "Complete Bending Experiment" described in para. 124 and Fig. 174, the load was applied to the beam from beneath by means of the straining head of the testing machine (an Amsler-Laffan Beam Testing Machine), which rose as the load was increased while the bearings of the beam at the ends remained stationary; these conditions are illustrated diagrammatically in Fig. 200.

To measure change in slope at the point C, immediately over one of the loads W, a small mirror resting on three points of support was placed on the top of the beam at K, immediately over C. A scale HG was then fixed at a considerable distance from the mirror, the distance KH (or M) in this case being 14·25 ft. A telescope was placed at E and the mirror tilted until, by reflection, it was possible to take a reading of the scale through the telescope, a cross-hair being fixed in the telescope for this purpose.

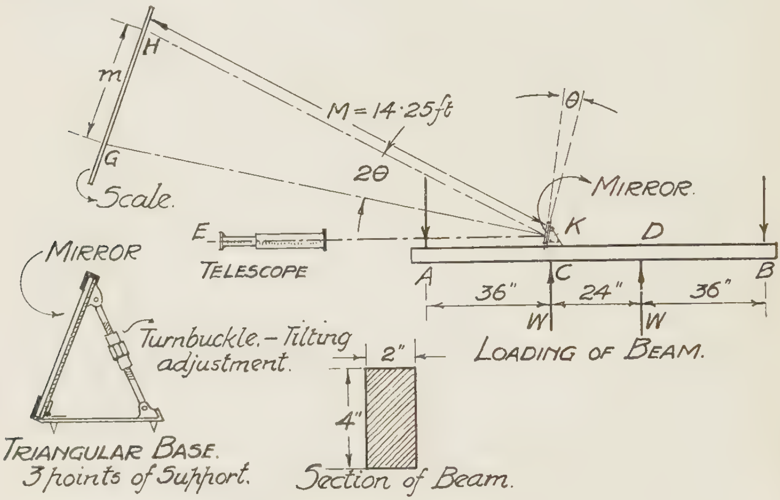


Fig. 200. Slope of a beam.

When the loads W, W are added—or better still when they are increased by a known amount—the beam bends and so becomes inclined at C, the mirror making the same change in inclination, say θ. It is possible to show that, if the mirror changes its inclination by an angle of θ, then, in accordance with a well-known law of the reflection of light, the reflected ray KH will change to KG, so that the angle HKG is equal to 2θ. Since the distance M is considerable, while the distance m on the scale is usually small, the ratio m/M will give the value of the angle 2θ in radians, or

$$\theta \text{ (radians)} = \text{change of slope} = \frac{m}{2M}. \qquad \ldots\ldots(a)$$

In the experiment described in para. 124 the following results were obtained, the mirror being placed over C; see Fig. 200.

Load W (lbs.) Increasing..................	1000	1500	2000	2500	3000
Change of reading on scale (m) in ins.	0·25	0·25	0·25	0·2	0·25
Load W (lbs.) Decreasing	2500	2000	1500	1000	
Change of reading on scale (m) in ins.	0·2	0·2	0·25	0·2	

Average change of reading for change of W = 500 lbs. is 0·225 in. Then, from (a) above,

$$\text{Change of slope} = \theta = \frac{m}{2M} = \frac{0{\cdot}225}{2 \times 14{\cdot}25 \times 12} = 0{\cdot}000658 \text{ radian.}$$

By calculation, since

$$L = 24 \text{ ins.},$$
$$B = 500 \times 36 = 18{,}000 \text{ lb. ins.},$$
$$I = 10{\cdot}67 \text{ inch units}^4,$$
$$E = 30{,}000{,}000 \text{ lbs. per sq. in. (see para. 124)},$$

we have $\theta = \dfrac{BL}{2EI} = \dfrac{18{,}000 \times 24}{2 \times 30{,}000{,}000 \times 10{\cdot}67} = 0{\cdot}000675$ radian.

145. Deflection of cantilevers. Although the case of deflection and slope dealt with in the two preceding paragraphs is of considerable importance, it is found necessary to approach the investigation of deflection and slope in nearly all other cases along somewhat different lines, since only in rare cases is a beam in practice subjected to a constant bending moment over any considerable portion of its length. The general principles underlying this method are given in the following statement.

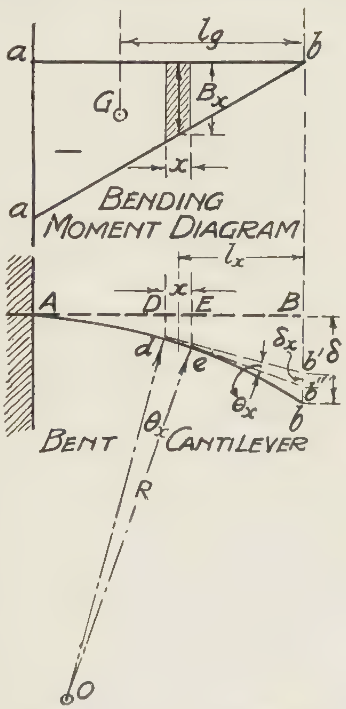


Fig. 201. Slope and deflection of a cantilever.

Deflection of a cantilever. (General case.) Let the line $ADEB$ in Fig. 201 represent a cantilever which, after loading in some manner (which need not for the moment be specified), bends to the curve $Adeb$, and let aab be the bending moment diagram for such loading. If two points d and e be taken on the curved outline of the beam, so that the distance x between them is extremely small, then we may assume that the radius of curvature is the same at each of these points and equal to R; see Fig. 201. (The distortion of the beam $Adeb$ is of course considerably exaggerated in Fig. 201.)

If tangents db' and eb'' be now drawn from d and e respectively, reaching to the ordinate Bb at the end of the cantilever, then *the distance* b′b″ *is the portion of the deflection at the end of the cantilever which is due to the change of slope between the two points* D *and* E. The magnitude of this change is given by the angle θ_x between these two tangents. But by the geometry of the figure

the angle *doe* between the radii at d and e is also equal to θ_x and, if θ_x is measured in radians, then

$$\theta_x = \frac{x}{R} = \frac{b'b''}{l_x},$$

where l_x is the distance from the end of the cantilever to the small portion *de*. If δ_x stands for the small distance $b'b''$, we see that

$$\delta_x = \theta_x \times l_x.$$

If now we desire to obtain δ, the total deflection at the end of the cantilever, it is clear that we shall get it by adding together all the small deflections such as δ_x, obtained in the manner described, from the changes in slope over the small lengths such as x. Now, as we have seen, $\theta_x = x/R$, and we also know that

$$\frac{1}{R} = \frac{B_x}{EI},$$

where B_x is the mean bending moment between d and e at a distance of l_x from B (see also para. 143). Hence we have

$$\theta_x = x\left(\frac{1}{R}\right) = \frac{B_x \times x}{EI};$$

but the numerator of this fraction ($B_x \times x$) is equal to the area of the shaded portion of the bending moment diagram, see Fig. 201. If this area is called A_x, then we have

$$\theta_x = \frac{A_x}{EI}. \qquad \ldots\ldots\text{(a)}$$

Similarly, since $\delta_x = \theta_x \times l_x$, we have

$$\delta_x = \frac{A_x \times l_x}{EI}. \qquad \ldots\ldots\text{(b)}$$

To find the total slope we must evidently sum the first expression (a) over the whole length of the beam, in which case A_x becomes the area of the bending moment diagram, so that, if this be A, we have

$$\textbf{Total slope at } \boldsymbol{B} = \boldsymbol{\theta} = \Sigma \frac{A_x}{EI}$$

$$= \frac{A}{EI} = \frac{\textbf{area of bending moment diagram}}{E \,.\, I}. \qquad \ldots\ldots\text{(iii)}$$

To find the total deflection at B we must obtain the sum of the second expression (b) over the length of the beam, so that

$$\text{Total deflection} = \delta = \Sigma \frac{A_x l_x}{EI}$$

$$= \frac{1}{EI} \Sigma \, (A_x \times l_x);$$

but, since l_x is the distance from B to the centroid of the small area A_x, the term $\Sigma\ (A_x l_x)$ must be equal to the first moment of the whole area (A) of the bending moment diagram about B, the free end of the cantilever. If G be the centroid of the bending moment diagram and the distance to G from B be l_g, then we have

$$\text{Total deflection at } B = \delta = \frac{Al_g}{EI}$$

$$= \frac{\text{first moment of B.M. diagram about free end of cantilever}}{EI}. \quad \ldots\ldots\text{(iv)}$$

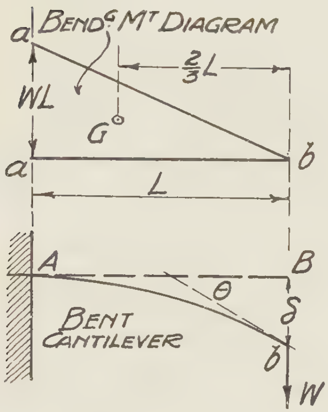


Fig. 202. Cantilever with end load.

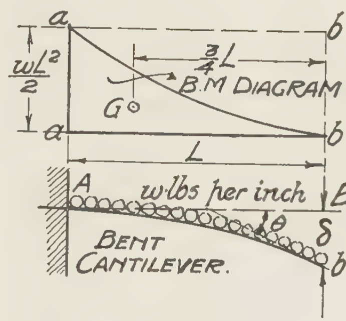


Fig. 203. Cantilever with uniformly distributed load.

146. Examples—Cantilevers. We may now proceed to apply these general expressions to particular cases.

(*a*) **Cantilever with end load W.** See Fig. 202.

In this case A, the area of the bending moment diagram,

$$= \frac{WL \times L}{2} = \frac{WL^2}{2}.$$

Also l_g, the distance from B to G, the centroid of the bending moment diagram, will be $2L/3$. Hence we have

$$\text{Total slope at } B = \theta = \frac{A}{EI} = \frac{WL^2}{2} \times \frac{1}{EI} = \frac{WL^2}{2EI}. \quad \ldots\ldots\text{(v)}$$

Similarly

$$\text{Total deflection at } B = \delta = \frac{Al_g}{EI} \quad \left(\frac{WL^2}{2} \times \frac{2L}{3}\right)\frac{1}{EI} = \frac{WL^3}{3EI}. \quad \ldots\ldots\text{(vi)}$$

(*b*) **Cantilever with uniformly distributed load w lbs. per in.** See Fig. 203. In this case the maximum bending moment is $wL^2/2$, and the outline of the diagram is parabolic. The area A may be shown to be

$$= \frac{1}{3}\,(\text{enclosing rectangle}) = \frac{1}{3}\left(L \times \frac{wL^2}{2}\right) = \frac{wL^3}{6}.$$

The distance from B to G, the centroid of the diagram, may be shown to be $\frac{3}{4}L$ in this case. Then

$$\textbf{Total slope at } B = \theta = \frac{A}{EI} = \left(\frac{wL^3}{6}\right)\frac{1}{EI} = \frac{wL^3}{6EI}. \qquad \text{......(vii)}$$

Similarly

$$\textbf{Total deflection at } B = \delta = \frac{Al_g}{EI} = \left(\frac{wL^3}{6} \times \frac{3L}{4}\right)\frac{1}{EI} = \frac{wL^4}{8EI},$$

or, if W be substituted for (wL), then

$$\delta = \frac{WL^3}{8EI}. \qquad \text{......(viii)}$$

(*c*) **Cantilevers with any loading.** With other types of loading the work proceeds on exactly the same lines as those given above. If the bending moment diagram is rather complicated it can usually be divided up into several simple figures, for which the areas and the distances from the free end of the cantilever to their centroids can easily be obtained. The quantity $(A \times l_g)$ will then be equal to the sum of the products of each pair of such quantities as $(A_x \times l_x)$. The following worked example will make this clear (see also para. 149).

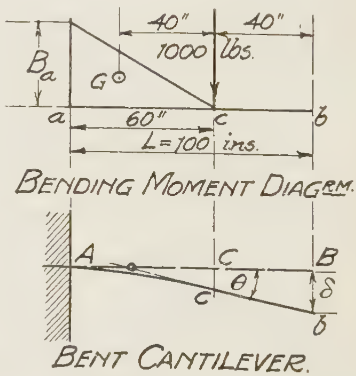


Fig. 204. Cantilever with load not at the end.

(*d*) **Worked Example.** *To find the slope and deflection at the end of a cantilever carrying a load at some other point than the end.*

We will use the dimensions and values given in Fig. 204 and assume that the value of E for the material of the cantilever is 30,000,000 lbs. per sq. in. and that I has a value of 150 inch units[4].

Then the maximum bending moment occurs at A and

$$B_a = 1000 \times 60$$
$$= 60{,}000 \text{ lb. ins.}$$

Between C and B the bending moment is zero. The area (A) of the bending moment diagram

$$= \frac{60{,}000 \text{ lb. ins.} \times 60 \text{ ins.}}{2}$$
$$= 1{,}800{,}000 \text{ lb. inch units}^2.$$

Also the distance from B to G, the centroid of the bending moment diagram, is $40 + (\frac{2}{3} \times 60)$ or 80 ins. Then slope at B

$$= \theta = \frac{A}{EI} = \frac{1{,}800{,}000}{30{,}000{,}000 \times 150} = 0{\cdot}0004 \text{ radian}.$$

(This can be brought to degrees by use of a table of radians or by multiplying by the constant 57·3.)

Similarly the deflection at B

$$= \delta = \frac{Al_g}{EI} = \frac{1{,}800{,}000 \times 80}{30{,}000{,}000 \times 150} = 0{\cdot}032 \text{ in.}$$

(*e*) **To calculate the deflection of a cantilever by means of the integral calculus.** To those readers who are acquainted with and are desirous of using the methods of the calculus, the following note will explain the slight modification necessary in the above methods to enable this to be done. Referring once more to Fig. 201, it is obvious that we may write down the small dimension x in terms of l, the distance measured along the cantilever, as dl. The expression

$$\delta_x = \frac{A_x l_x}{EI} = \frac{(B_x x)\, l_x}{EI}$$

then becomes

$$\delta_x = \frac{B_x \times dl \times l_x}{EI}.$$

Summing these small quantities over the whole length of the beam, we therefore write

$$\delta = \text{total deflection} = \frac{1}{EI}\int_{l=0}^{l=L} B\,.\,l\,.\,dl,$$

where, in the limit, B is the bending moment at A (where the slope is zero; see note in para. 148), the point A being at a distance L from B.

Example. In the case of a cantilever with an end load W, where $B = Wl$, the above expression is then written

$$\delta = \frac{W}{EI}\int_{l=0}^{l=L} l^2\,.\,dl,$$

which

$$= \frac{WL^3}{3EI},$$

as in (vi) in this paragraph.

Other cases are dealt with similarly.

147. Deflection of freely supported beams. If for the present we limit our consideration of deflection to those cases of beams which are symmetrically loaded about their centre point, then it is clear that the maximum deflection will occur at the centre section. Thus in Fig. 205, where we have a beam carrying a load W at the centre C, and also in Fig. 206, where we have a beam carrying a uniformly distributed load over its entire length, the distance CD gives the maximum deflection (δ). In addition the shape of the bent beam must obviously be symmetrically disposed about the centre line at C, and it follows from this, that if we draw a

tangent to the curve at D, then this tangent will be parallel to the original line through AB, the points of support. In other words *the slope at the centre of the bent beam in both these cases is zero.*

Hence it will be seen that, if we reverse the figure of the bent beam, as has been done in Fig. 205, we may look upon it as being made up of two cantilevers, starting with zero slope at D' in each case, and extending for a distance $(L/2)$ to A' and B' respectively. In each case the deflection at the end of each "cantilever" is equal to the deflection δ at the centre of the beam. When viewed in this way, the deflection at the ends of these imaginary cantilevers is evidently due to the reactions at A and B, which in both these cases is equal to half the total load. Utilising these relations, we are therefore able to arrive at the magnitude of the deflection in each of these cases.

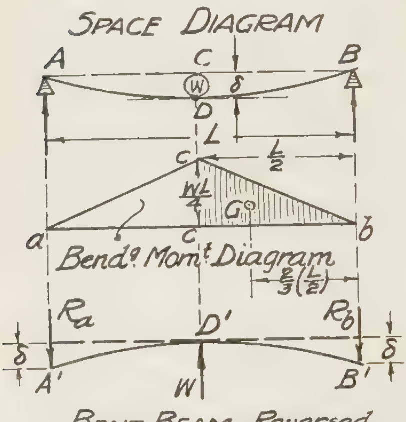


Fig. 205. Beam with load at centre.

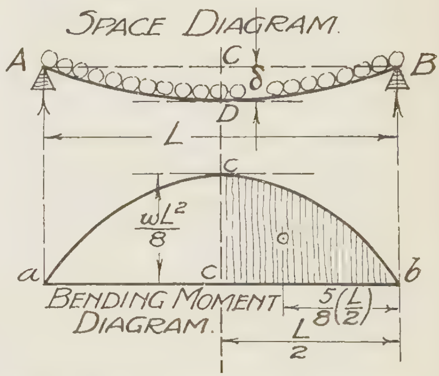


Fig. 206. Beam with uniformly distributed load.

(*a*) Deflection of a beam loaded at the centre. See Fig. 205. Consider the portion $D'B'$ as a cantilever with a load $W/2$ at the end B', then the bending moment diagram is given by the triangle ccb, so that A, the area of this figure,

$$= \frac{WL}{4} \times \frac{L}{2} \times \frac{1}{2} = \frac{WL^2}{16}.$$

Also the distance (l_g) from B to G, the centroid of this figure, is $\frac{2}{3}(L/2)$, or $L/3$. Hence we have

$$\delta = \text{deflection of beam} = \frac{Al_g}{EI} = \frac{WL^2}{16} \times \frac{L}{3} \times \frac{1}{EI} = \frac{WL^3}{48EI}. \quad \text{......(ix)}$$

(*b*) Deflection of a beam with uniformly distributed load. See Fig. 206. In this case the outline of the bending moment diagram is parabolic, the maximum ordinate being equal to $wL^2/8$. Again, reckoning DB as

a cantilever, the portion of the bending moment diagram with which we are concerned is the figure *ccb*. The area (A) of this is

$$= \frac{2}{3}\left(\frac{wL^2}{8} \times \frac{L}{2}\right) = \frac{wL^3}{24}.$$

Also the distance (l_g) from B to the centroid of this figure may be shown to be

$$= \frac{5}{8}\left(\frac{L}{2}\right) \text{ or } \frac{5L}{16}.$$

Hence, from $\delta = Al_g/EI$, we have

$$\delta = \frac{wL^3}{24} \times \frac{5L}{16} \times \frac{1}{EI} = \frac{5wL^4}{384EI},$$

or, if W be substituted for wL,

$$\delta = \frac{5WL^3}{384EI}. \qquad \text{......(x)}$$

148. Conditions of application of the general expressions given in para. 145 for the slope and deflection of a cantilever. If reference be made to Fig. 201 it will be seen that the line AB, which represented the unstrained position of the cantilever, might be looked upon as a base line from which both slope and deflection are measured. It will thus be seen that both slope and deflection are zero at A. This latter point might therefore be called "the point of origin". It follows that the general expressions for slope and deflection which were obtained in para. 145 can be applied to any case of a beam or cantilever provided that A, the point of origin, is taken at some point at which the slope is either zero or—which is the same thing—at which the slope and deflection of the beam are known. (In the latter case the calculated slope and deflection are merely to be added to that already known.)

Further, it should be noted that the point B need not be at the end of the beam, and that we can, in fact, calculate the slope and deflection at any point between A and B from the general expressions.

Example. *To find the slope at the point* C *in the cantilever shown in* Fig. 204.

Ignoring the portion CB of the cantilever beyond the point C we find the slope at C by means of the expression $\theta = A/EI$, *where* A *is the area of the bending moment diagram between* A *and* C. But A in this case will have exactly the same value as before, so that, the other terms E and I being the same, the slope at C is the same as at B. In other words the portion *cb* of the strained cantilever is straight.

149. Experiment. *To find the deflection and slope of a beam with two-point loading.* See Fig. 207.

(The values used below were obtained in the Complete Bending Experiment described in para. 124, the dimensions of the beam and loading given in Fig. 207 being the same.)

Deflections. These were measured by means of a micrometer length gauge; see Fig. 207. A stout straight-edge (not shown) was placed upon the points of support of the beam, measurements being made from this straight-edge to points C, D and E on the beam, as the test proceeded. A large number of readings were taken for equal increases of the loads W (500 lbs. at a time being added), and the figures given below are the averages of at least ten such readings.

Experimental readings. Deflection at C (and D) for a change of the loads W of 500 lbs. each = 0·0475 in.

Deflection at E for a change of the loads W of 500 lbs. each = 0·0549 in

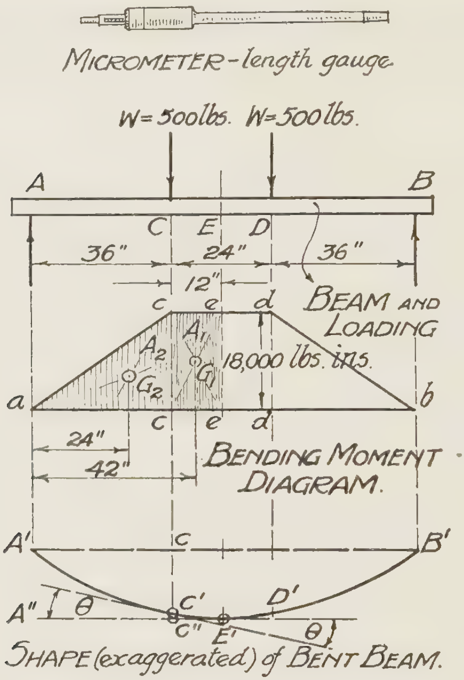


Fig. 207. Slope and deflection of a beam with two-point loading.

Calculated deflections. See Fig. 207. Following the methods outlined in this chapter, the total deflection of the beam at E' will equal the total deflections of the imaginary "cantilevers" at A' and B'.

To find the deflection of the "cantilever" ECA. The bending moment diagram is as shown, the constant bending moment between C and D being equal to (500 × 36) or 18,000 lb. ins. Divide this up into two portions A_1 and A_2 as shown, between the points a and e.

Then the total deflection at A

$$= A'A'' = \frac{Al_g}{EI} = \frac{(A_1 \times 42) + (A_2 \times 24)}{EI\,.};$$

see (c), para. 146.

Now the "area" of $A_1 = (12 \times 18{,}000)$ lb. ins.^{2}
While the "area" of $A_2 = (36 \times 18{,}000 \times \frac{1}{2})$ lb. ins.^{2}

$$E = 30{,}000{,}000 \text{ lbs. per sq. in.}$$

$$I = \frac{bd^3}{12} = \frac{2 \times 4^3}{12} = 10{\cdot}67 \text{ inch units}^4.$$

Then the deflection at $A' = \delta =$ deflection of beam at E'

$$= \frac{(18{,}000 \times 12 \times 42) + (18{,}000 \times 18 \times 24)}{30{,}000{,}000 \times 10{\cdot}67} = 0{\cdot}0526 \text{ in.}$$

To find the deflection of the beam at C. The total deflection at E' is equal to the distance CC''; see Fig. 207. The distance $C'C''$ is equal to the deflection of the "cantilever" at C, taking E' as the "point of origin." The deflection of the beam at C is therefore equal to $(CC'' - C'C'')$.

Deflection of the "cantilever" at C

$$= C'C'' = \frac{A_1 \times 6}{EI} = \frac{(18{,}000 \times 12)\, 6}{30{,}000{,}000 \times 10{\cdot}67} = 0{\cdot}00405.$$

Then deflection of C on the beam

$$= CC'' - C'C'' = 0{\cdot}0526 - 0{\cdot}00405 = 0{\cdot}04855 \text{ in.}$$

To find the slope at C. The manner in which this was done experimentally has been explained in para. 144, the result obtained being 0·000658 radian. As an additional check we may calculate the slope (θ) from the area of the bending moment diagram (A_1). Slope at C

$$= \theta = \frac{A_1}{EI} = \frac{18{,}000 \times 12}{30{,}000{,}000 \times 10{\cdot}67} = 0{\cdot}000675 \text{ (radian).}$$

150. Strength and stiffness. In Chap. XIV, where we dealt with the problem of ascertaining the dimensions of beam sections which were necessary to keep the maximum stresses within safe limits, we were really dealing with the **Strength** of the beams. In designing beams to be used in buildings, however, it is frequently of importance to ensure that the beams are also stiff, and do not bend or deflect to an excessive extent; otherwise cracks may occur in the walls which are carried by these beams, or in the plasterwork or other enrichments which may be attached thereto, or excessive vibrations may be set up in floors when subjected to moving loads.

The term **Stiffness** is thus used to indicate the resistance of a beam against deflection. In order that comparisons may be made between beams of varying dimensions, stiffness is usually expressed in terms of the deflection and span of the beam, so that

$$\text{Stiffness} = \frac{\textbf{deflection}}{\textbf{span}}. \qquad \text{......(xi)}$$

In ordinary building work this ratio is commonly limited to $\frac{1}{400}$, but the fraction may be smaller in the case of large and important beams.

If the various expressions used to find deflection be compared, it will be seen that they are of the form

$$\text{Deflection} = K \cdot \frac{WL^3}{EI},$$

where K is a constant which varies with the manner of loading and supporting the beam (see next chapter and also Table VI). If we consider beams of rectangular section, and substitute $(bd^3/12)$ for I in the above expression, we have

$$\text{Deflection} = K \cdot \frac{12WL^3}{Ebd^3},$$

from which we see that *the deflection of a rectangular beam varies directly as the load and as the cube of the span, and inversely as the breadth and the cube of the depth.* (This statement should be compared with that given in para. 133.) The importance of the vertical dimension of the beam is thus seen to be considerable if stiffness is desired.

Problems XVI

1. Calculate the upward deflection at point E in the beam of Fig. 199, if $L = 10$ ft.; $L_1 = 3$ ft.; $W = 200$ lbs.; $E = 1{,}500{,}000$ lbs. per sq. in., the beam being of timber; the breadth $(b) = 2$ ins., and the depth $(d) = 4$ ins.

2. Calculate the total change of slope at A in the beam described in Prob. 1; see also Fig. 199.

3. Find the change in the value of W in Fig. 200 if, when $M = 15$ ft., the alteration in the scale reading (m) is 2 ins. Use the same values for E (30,000,000) and I (10·67).

4. A main floor beam AB carries two equal loads of 5 tons at points C and D, which are each 8 ft. from the ends of the beam. The total span AB is 30 ft. (Fig. 207 may be used, the lettering being the same.) Calculate the slope in the beam at A and C due to the stated loads, the beam being originally straight and horizontal. Take $E = 30{,}000{,}000$ lbs. per sq. in. and $I = 1200$ inch units^4.

5. Calculate the deflections at points C and E in the beam in Prob. 4. Express the deflection at E as a ratio of the span of the beam AB.

6. A cantilever 10 ft. long carries a distributed load of 0·5 ton per ft. run throughout its length, and a concentrated load of 2 tons at 5 ft. from the free end. Calculate the slope and deflection at the free end of the cantilever, if $E = 30{,}000{,}000$ lbs. per sq. in. and $I = 490$ inch units^4.

7. A beam of uniform section, for which I has the value of 280 inch units^4, carries a uniformly distributed load of 0·25 ton per ft. run throughout its length of 15 ft. It also carries a concentrated load of 2 tons at the centre of the span. Calculate the total deflection of the beam at the centre, if $E = 30{,}000{,}000$ lbs. per sq. in.

CHAPTER XVII

FIXED AND CONTINUOUS BEAMS

151. Fixed Beams. In dealing with the cantilever shown in Fig. 201, we assumed that the end at A was so secured or fixed that, even when the cantilever was loaded, it remained horizontal at A, that is the slope at A was zero. When the end of a cantilever or beam is so secured it is known as a **Fixed End.** In actual practice the fixing can never be so perfect as that here defined, but, as the difficulties of allowing for fixings which are less than perfect are so great that they would carry our investigations far beyond the limits which we have set ourselves in this volume, we will continue to assume that at a fixed end the slope remains at zero after loading.

A beam with fixed ends. It follows from the above statement that a beam with fixed ends, or, more briefly, a **fixed beam, is one in which the ends remain at zero slope when the beam is loaded.** We will first consider the case of a beam in which this condition is produced in some other way than by "fixing".

Let the beam AB shown in Fig. 208 (a) have its ends prolonged by equal amounts (L_1) to C and D respectively. If now a load W is placed at E, the centre of AB, then the beam will bend as shown in Fig. 208 (b) and a slope of θ_1 will be produced at B. If A_1 is the area of the shaded portion *eeb* of the bending moment diagram, we know from our previous work that the slope at B

$$=\theta_1=\frac{A_1}{EI}. \qquad \text{......(a)}$$

We will call the complete bending moment diagram *aebe* the **Free Moment Diagram.**

It will be clear, however, that, by adding equal loads (W_1) at the points C and D, the ends of the beam could be bent down until, as shown in Fig. 208 (c), the slope of the beam both at A and B was zero. This result has evidently been achieved by applying at A and B moments of opposite sign to those produced by the load W.

But let us for a moment consider what effect the loads W_1 at C and D would have upon the shape of the beam if acting alone. As we already know, they would produce a negative and constant bending moment between A and B, the magnitude of which would

be $-W_1L_1$. (The complete bending moment diagram would be as shown in Fig. 208 (d), but we can disregard the dotted portions beyond A and B, since we are not at present concerned with the manner in which the bending moment diagram *between* A *and* B is produced but only with its effects.) The portion of the diagram between A and B we will call the **Fixing Moment Diagram.**

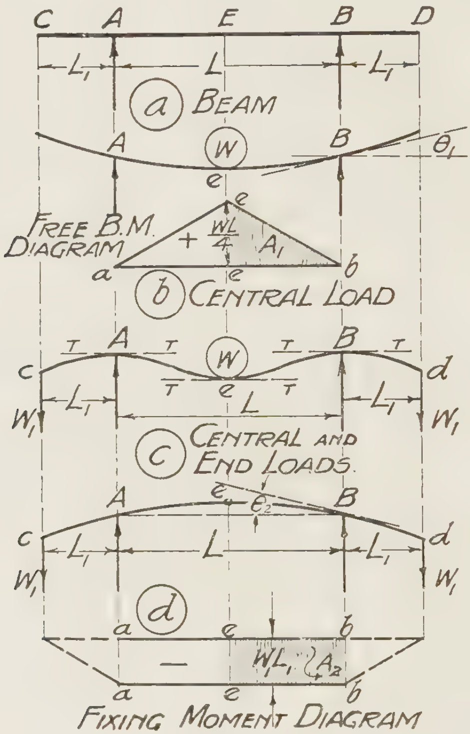


Fig. 208.

The shape which the beam would assume in the latter case is shown in Fig. 208 (d), the curvature between A and B being of an opposite kind to that produced by the central load W. If θ_2 be the slope produced in this case at B, it will thus be of opposite sign to that produced at B by the central load W.

As already explained we can find θ_2 by considering the portion eB of the bent beam as a "cantilever" fixed at e, for which the bending moment diagram is given by the shaded portion *eebb* of

the fixing moment diagram. If A_2 is the area of this portion, then we have, as before, the slope at B

$$= \theta_2 = \frac{A_2}{EI}. \qquad \text{......(b)}$$

If now we imagine all the loads acting as shown in Fig. 208 (c), then, in order that the "combined slope" at B may be zero, the slope (θ_2) produced at B by the load W_1 must just be equal to the corresponding slope (θ_1) produced by the central load W; so that we have

$$\theta_1 = \theta_2,$$

that is, from (a) and (b),

$$\frac{A_1}{EI} = \frac{A_2}{EI} \text{ or } A_1 = A_2.$$

But, since the loading is symmetrical, it is clear that in each case A_1 and A_2 represents half the complete bending moment diagram. Hence, calling AB a fixed beam, since its slopes at A and B are zero after loading, we have that:

I. In a fixed beam the free and fixing moment diagrams will be equal in area and opposite in sign.

(It should be noted that we have restricted our discussion to *symmetrically loaded beams*; it can be shown, however, that statement I is also true for beams which are not so loaded; in such cases other relations between the bending moment diagrams must then be known before the complete figure can be constructed.)*

To find the magnitude of the fixing moment in cases of symmetrical loading, since the shape of the fixing moment diagram is rectangular, it follows that its height may be obtained from the following relation:

$$\textbf{Height of fixing moment diagram} = \frac{\textbf{area of free moment diagram}}{\textbf{span}}. \qquad \text{......(i)}$$

Examples. The following examples will explain the application of these principles.

(a) **Fixed beam with central load.** Let AB in Fig. 209 be the fixed beam carrying a central load W at C.

Shear Force Diagram. Since we are not here concerned with the effect of the end fixing-moments upon the beam *outside of the length* AB, we may consider the two equal end fixing-moments (B_f) to be opposite couples in equilibrium; see Fig. 209 (c). These applied couples will add no vertical force, hence the shear force diagram must be the same as for

* For a fuller treatment of such problems the reader is referred to Morley's *Strength of Materials*, Andrews' *Theory and Design of Structures*, or Goodman's *Mechanics Applied to Engineering*.

a freely supported beam, the two vertical reactions at A and B being each equal to half the total load (W); see Fig. 209 (a).

Free Moment Diagram. This is shown at (b), the maximum bending moment occurring at the centre and being equal to $WL/4$. If A_f be the area of this free bending moment diagram, then

$$A_f = \frac{WL}{4} \times L \times \frac{1}{2} = \frac{WL^2}{8}.$$

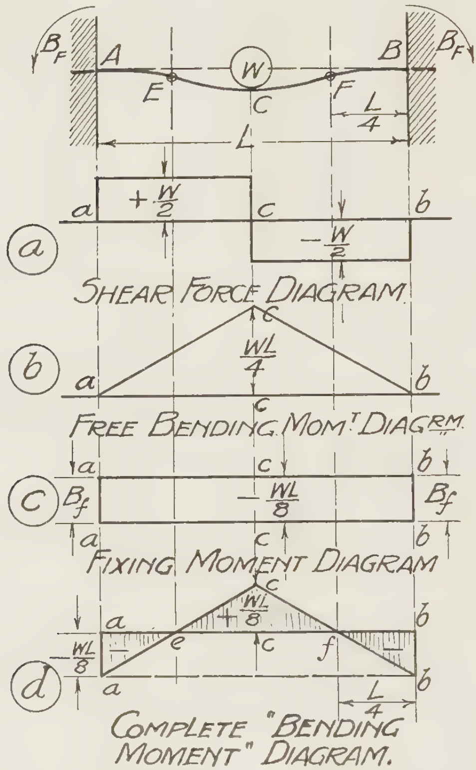


Fig. 209. Fixed beam with central load.

Fixing Moment Diagram. This is shown at (c), being rectangular and equal in area to A_f. Then if B_f be the fixing moment at A (and B) we have, from (i) above,

$$B_f = \text{fixing moment} = \frac{\frac{WL^2}{8}}{L} = \frac{WL}{8},$$

and is negative.

Complete Bending Moment Diagram. This is obtained by superimposing the free and fixing moment diagrams and omitting those parts

which overlap. From this it will be seen that, from a negative value of $-WL/8$ at each end, the bending moments pass through zero values at e and f to a maximum positive value of $WL/8$ at C, the centre of the beam.

Points of zero bending moment and contraflexure. As we have seen, there are two points, E and F, on the beam in Fig. 209 at which the bending moment is zero, E and F corresponding to the points e and f on the bending moment diagram. Evidently the beam at these points is not bent; they are, in fact, the points at which the shape of the beam is passing from a curve of one sign to a curve of the opposite sign, and they are therefore called *Points of Contraflexure.* It is frequently important to know where such points occur. In the case of the beam just dealt with, it is easy to see (from symmetry) that they occur at points which are one-quarter of the span from each end; see Fig. 209 (*d*).

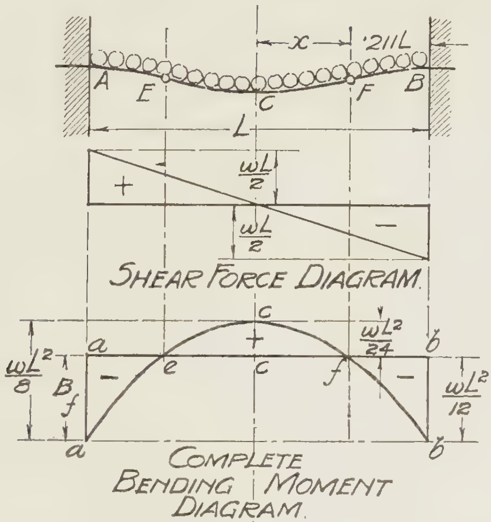


Fig. 210. Fixed beam with uniformly distributed load.

(*b*) **Fixed beam with uniformly distributed load.** See Fig. 210. In this case the maximum bending moment for the beam freely supported is $wL^2/8$, where w is the load per unit of length of the beam. The shape of the free moment diagram being parabolic, then its area (A_f) will be given by

$$A_f = \frac{wL^2}{8} \times L \times \frac{2}{3} = \frac{wL^3}{12}.$$

The area of the fixing moment diagram will also be equal to A_f, so that B_f, the negative fixing moment,

$$= \frac{A_f}{L} = \frac{wL^2}{12}.$$

With these values the complete bending moment diagram may be drawn; see Fig. 210. From this diagram we can see that the maximum

positive bending moment, which occurs at C, the centre of the beam, is equal to

$$\frac{wL^2}{8} - \frac{wL^2}{12} = \frac{wL^2}{24}.$$

The points of contraflexure will occur at points e and f, where the bending moments are zero, that is where the ordinates of the free bending moment diagram are equal to those of the fixing moment diagram, that is to $wL^2/12$.

To find the positions of these points, let x be the distance from C to F, then the bending moment at F will be given by

$$\frac{wL^2}{8} - \left(wx \times \frac{x}{2}\right),$$

which $$= \frac{w}{2}\left(\frac{L^2}{4} - x^2\right).$$

If this is to equal $wL^2/12$, the magnitude of the fixing moment at F, then

$$\frac{w}{2}\left(\frac{L^2}{4} - x^2\right) = \frac{wL^2}{12},$$

or $$\frac{L^2}{4} - x^2 = \frac{L^2}{6},$$

whence $$x^2 = \frac{L^2}{4} - \frac{L^2}{6} = \frac{L^2}{12},$$

or $$x = \frac{L}{\sqrt{12}} = \frac{L}{2\sqrt{3}}$$
$$= 0{\cdot}289L.$$

Measuring from the end of the beam at A (or B) the distance to the point of contraflexure is $0{\cdot}211L$, see Fig. 210.

152. The effect of imperfect fixing. If, by reason of the manner in which a beam is fixed, the slopes at the ends do not remain at zero, then it is not difficult to see that the "fixing moment" (B_f) will not reach the full value given in the expressions in Examples (a) and (b), para. 151; the base line *aefb* of the bending moment diagrams will therefore be lowered and the values of all the negative bending moments reduced. The values of the positive bending moments will, of course, be correspondingly increased.

In some cases the magnitude of the fixing moment (B_f) which may be used is specified by regulation, or it may be estimated from a knowledge of the construction of the building and the character of the connections at the ends of the beam. The complete bending moment diagram can then be obtained by a suitable modification of the diagram for perfect fixing.

153. Deflection of fixed beams. The simplest method for finding the deflection of a fixed beam is to look upon it as the difference between the deflection (δ_s) of a similarly loaded but freely supported

beam, and the deflection (δ_f) due to the fixing bending moment (B_f). We will deal only with two cases.

(*a*) **Deflection of a centrally loaded fixed beam.** See Fig. 211. In this case δ_s, the deflection in the case of the freely supported beam, will be given by

$$\delta_s = \frac{WL^3}{48EI}, \text{ see (ix), para. 147.} \qquad \ldots\ldots\text{(a)}$$

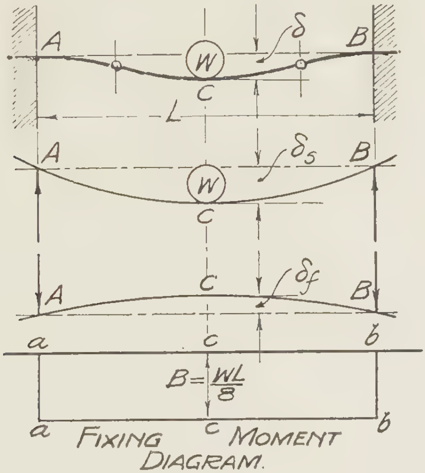


Fig. 211. Deflection of fixed beam with central load.

Also δ_f, the deflection of the beam in an opposite direction when acted upon by a uniform fixing moment of $-WL/8$, see Fig. 211, will be given by

$$\delta_f = \frac{BL^2}{8EI},$$

see (i), para. 143, where B is equal to the fixing moment which, in this case, is $WL/8$. Substituting this value for B, we have

$$\delta_f = \frac{WL}{8} \times \frac{L^2}{8EI} = \frac{WL^3}{64EI}. \qquad \ldots\ldots\text{(b)}$$

Then, from (a) and (b), the deflection of a fixed beam with a central load

$$\delta = \delta_s - \delta_f = \frac{WL^3}{48EI} - \frac{WL^3}{64EI} = \frac{WL^3}{192EI}, \qquad \ldots\ldots\text{(ii)}$$

which is one-quarter of the deflection of a similarly loaded but freely supported beam.

(*b*) **Deflection of a fixed beam with uniformly distributed load.** See Fig. 212. In this case the deflection of the beam when freely supported

$$= \delta_s = \frac{5wL^4}{384EI}.$$

Also, since the fixing moment in this case is $wL^2/12$, see Fig. 210, we can substitute this value for B in the expression

$$\delta_f = \frac{BL^2}{8EI},$$

whence

$$\delta_f = \frac{wL^2}{12} \times \frac{L^2}{8EI} = \frac{wL^4}{96EI}.$$

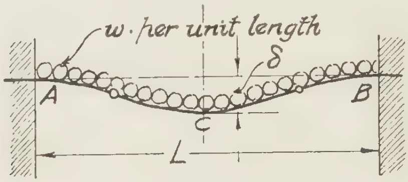


Fig. 212. Deflection of fixed beam with uniformly distributed load.

Whence δ, the deflection of a fixed beam with uniformly distributed load,

$$\frac{5wL^4}{384EI} - \frac{4wL^4}{384EI} = \frac{wL^4}{384EI} = \frac{WL^3}{384EI} \quad \text{......(iii)}$$

(by substituting W for wL), and this is only one-fifth of the deflection which the same load would produce if the beam was freely supported.

154. Continuous Beams. When a beam is supported at more than two points it is known as a **Continuous Beam.** To investigate the subject of continuous beams with any fullness would be impracticable within the limits of this volume. Since, however, by a simple extension of the work which we have already done, we may deal with a few simple cases of symmetrical loading, we will do so in order to get some notion as to how shearing forces and bending moments are distributed in such beams.

Let the beam ACB in Fig. 213 be of uniform section throughout its length, and supported at three equidistant points A, B and C, these points being assumed to be on the same level, *both before and after the beam is loaded.* The span in each case equals L_1.

Reactions. If there were no support at C, then we could find the deflection (δ_c downwards) at C, for the uniformly distributed load, from the expression (x), para. 147, when

$$\delta_c \text{ (downwards)} = \frac{5wL^4}{384EI},$$

where $L = 2L_1$. Then, since $(2L_1)^4 = 16L_1{}^4$, we have

$$\delta_c \text{ (downwards)} = \frac{80wL_1{}^4}{384EI}. \quad \text{......(a)}$$

Now it will be evident to the reader that the force (R_c), which is the reaction at C, must be just sufficient to remove this deflection δ_c so as to restore the point C to the level of A and B. Hence, if we imagine an upward force equal to R_c acting at the support C,

then the deflection δ_c (upwards) which it will produce over the full span of $2L_1$ must be equal to the deflection δ_c (downwards) which we have just obtained. From this relation the magnitude of R_c can be found. For a central load we have

$$\delta_c \text{ (upwards)} = \frac{WL^3}{48EI}, \text{ see (ix), para. 147,}$$

where $W = R_c$, and $L = 2L_1$; hence we have

$$\delta_c \text{ (upwards)} = \frac{8R_cL_1^3}{48EI}. \qquad \text{......(b)}$$

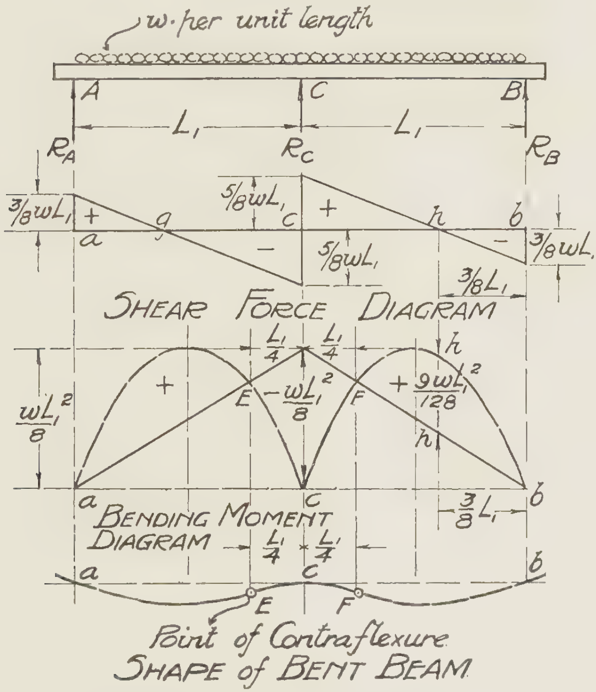


Fig. 213. Continuous beam.

But since δ_c (upwards) = δ_c (downwards), then (a) and (b) must be equal, so that

$$\frac{8R_cL_1^3}{48EI} = \frac{80wL_1^4}{384EI},$$

or

$$R_cL_1^3 = \frac{10wL_1^4}{8},$$

or

$$R_c = \text{Reaction at } C = \frac{5wL_1}{4}.$$

Since the total load on the beam is equal to $2wL_1$, and this is symmetrically distributed, it follows that R_a must equal R_b and each must equal $\frac{1}{2}(2wL_1 - 5wL_1/4)$, from which we have that

$$R_a = R_b = \frac{3wL_1}{8}.$$

Shear Force Diagram. With these values we are able to complete the shearing force diagram; see Fig. 213. At C, since half the total reaction is due to each part of the beam, the total reaction is divided into two equal parts as shown. The rate at which the values fall off is of course the same over each span, and no difficulty should be experienced in constructing the diagram in order to show the alternating positive and negative portions.

Bending Moment Diagram. To construct the bending moment diagram, first draw what may now be called the "free moment diagrams", showing the magnitude of the bending moments over each span as if the beams were separate and freely supported. The maximum ordinate will of course occur at the centre of each span and be equal to $wL_1^2/8$; see Fig. 213.

If AC and CB had been separate beams, then the outer reactions at A and B would have been equal to $wL_1/2$, but this is reduced in the continuous beam to $3wL_1/8$, so that the outer supports are relieved in this way to the extent of $wL_1/8$. This reduction is evidently due to the lifting effect of R_c, from which it follows that there must be a *negative* "*fixing moment*" at C equal to $[(wL_1/8)\,L_1]$. Hence the fixing moment at C is $wL_1^2/8$, which is the same value as for the maximum ordinates of the free moment diagrams. Superimposing the two diagrams we get the complete bending moment diagram shown in Fig. 213.

It is easy to show that the outline of the shear force diagram will cut the base line acb at points g and h, which are $3L_1/8$ from A and B respectively. These being the points at which the shear force is zero, they will also be the points at which the bending moments reach a maximum; see para. 92, III. By calculation this may be shown to be equal to $9wL_1^2/128$.

In the bent beam the points of contraflexure E and F, at which the bending moments are zero, may be shown to occur at $L_1/4$ from the centre point C; see Fig. 213.

The above methods may be applied to all cases of continuous beams which are supported at three points and loaded symmetrically about the centre point.

155. The "Propped Cantilever". If the cantilever CB in Fig. 214 be supported or "propped" at B, so that B and C remain level with

each other before and after loading, then we may find the reactions at C and B and construct the bending moment diagram by an application of the above methods.

Since the cantilever is fixed at C, the slope remains at zero after loading, and we may imagine the cantilever extended to A, an equal distance L_1 to the left of C, so that ACB forms an imaginary continuous beam, symmetrically loaded about C and uniform in section throughout its length. Considered in this

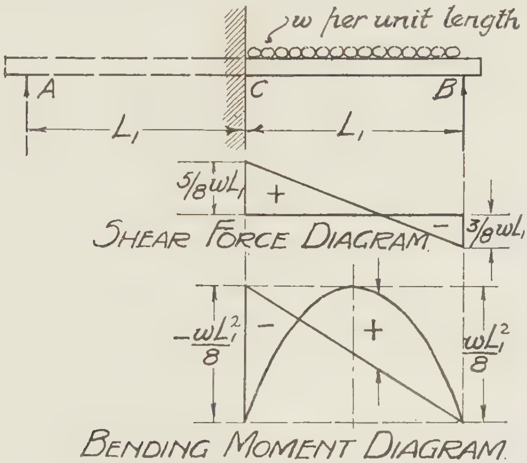


Fig. 214. The "propped cantilever".

way it is easy to see that the shear force diagram and the bending moment diagram for the cantilever will correspond to the half diagrams for the complete continuous beam; see Fig. 214. Cantilevers loaded in other ways and propped at other points may be dealt with in a similar manner.

Problems XVII

1. If in Fig. 208 the distance $L = 12$ ft. and $L_1 = 3$ ft., find the magnitude of W_1 in terms of W so that the slopes at A and B are both zero when the beam is subjected to the three loads W, W_1 and W_1.

2. Draw the shear force and bending moment diagrams for a centrally loaded fixed beam, as in Fig. 209, when $L = 20$ ft. and $W = 5$ tons. State the magnitude of the maximum positive and negative bending moments.

3. Find the central deflection for the beam in Prob. 2 if $I = 150$ inch units and $E = 30,000,000$ lbs. per sq. in.

4. Draw the shear force and bending moment diagrams for a fixed beam carrying a uniformly distributed load of $\frac{1}{4}$ ton per ft. run over a span of 12 ft. State the maximum positive and negative bending moments.

5. Find a suitable value of I for the beam in Prob. 4 if the deflection at the centre of the beam must be limited to 1/400 span, and $E = 30{,}000{,}000$ lbs. per sq. in. Select a suitable section from the Table given in Appendix I.

6. Draw the shear force and bending moment diagrams for a beam with two-point loading as described in para. 144 and illustrated in Fig. 200, using the same dimensions for the span of the beam, position of the two equal loads (W), and the dimensions of the beam, but assuming that the ends A and B of the beam are fixed. State the magnitudes of the fixing-moments at A and B, and of the bending moments between the points C and D, when the loads each equal 3000 lbs. (Note. Apply statements I and (i) from para. 151.)

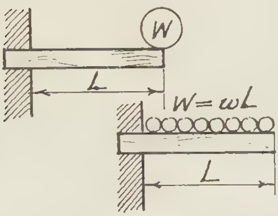
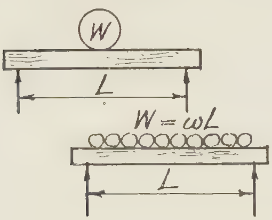
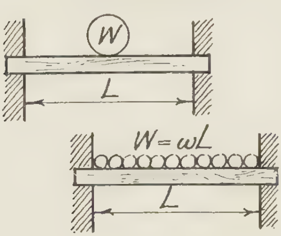
7. Draw shear force and bending moment diagrams for the continuous beam shown in Fig. 213, if $L_1 = 12$ ft. and $w = 200$ lbs. per ft. run. Give the magnitudes of the reactions at each point of support, and of the bending moments at C and at the point of maximum bending moment over each span.

8. If an I-section girder is to be used for the beam in Prob. 7, for which the flexural stress (f) is to be limited to $7\frac{1}{2}$ tons per sq. in., find, for the loading given in Prob. 7, the value of I and choose a suitable section from the Table given in Appendix I. The value of E may be taken at 30,000,000 lbs. per sq. in.

9. If the cantilever shown in Fig. 214 supports a gallery floor, for which $w = 1000$ lbs. per ft. run and $L_1 = 12$ ft., draw the shear force and bending moment diagrams for the cantilever. State the load carried by the pillar at B and the bending moment occurring at C. Also find the position and magnitude of the maximum bending moment occurring between C and B.

Table VI

Maximum Shear, Bending Moment and Deflection in Beams

Diagram of loading and fixing	Shear	Bending moment	Deflection
Cantilevers 	W	WL	$\frac{1}{3}\frac{WL^3}{EI}$
	W	$\frac{1}{2}WL$	$\frac{1}{8}\frac{WL^3}{EI}$
Freely supported Beams 	$\frac{1}{2}W$	$\frac{1}{4}WL$	$\frac{1}{48}\frac{WL^3}{EI}$
	$\frac{1}{2}W$	$\frac{1}{8}WL$	$\frac{5}{384}\frac{WL^3}{EI}$
Fixed Beams 	$\frac{1}{2}W$	$\frac{1}{8}WL$ (centre) $-\frac{1}{8}WL$ (ends)	$\frac{1}{192}\frac{WL^3}{EI}$
	$\frac{1}{2}W$	$\frac{1}{24}WL$ (centre) $-\frac{1}{12}WL$ (ends)	$\frac{1}{384}\frac{WL^3}{EI}$

SECTION III

AN INTRODUCTION TO SOME IMPORTANT STRUCTURAL PROBLEMS

CHAPTER XVIII

RIVETED CONNECTIONS. BUILT-UP BEAMS

156. Introduction to Section III. We have now completed, in Sections I and II, an outline of some of the more fundamental principles of Mechanics, Elasticity and the Strength of Materials, as applied in the analysis of systems of forces acting in framed structures and in solid beams. Though we have taken the opportunity from time to time to illustrate these principles by means of some familiar applications of the work in buildings, our main concern has been to present a simple and consistent statement, such as may serve as a sound foundation for further studies.

We have now reached a stage when, unless we propose to pursue the study of these subjects in their more advanced and theoretical aspects, the knowledge so gained may be extended and the fundamental principles further illustrated by a consideration of certain important applications peculiar to building operations and building construction. In any case a knowledge of such applications is of vital importance to the student of modern construction and, having acquired a knowledge of the fundamental principles of the subject of mechanics, he should be able to prosecute this study with a better appreciation of the significance of the methods which characterise modern construction, and of the changes which are introduced therein from time to time. The methodical study of full-size and large scale construction is, in fact, as important to the student and to the scientific investigator, as are the investigations of the classroom and the laboratory.

The order in which the remaining chapters have been placed has been determined to some extent by the difficulty of the topics, or by the amount of practical knowledge and experience which it is necessary to assume that the reader possesses. This order should not therefore be departed from except for adequate reasons. The order is as follows: riveted connections and built-up beams;

reinforced concrete beams; masonry construction and arches; compression members or columns in steel and reinforced concrete; timber construction.

157. Riveted connections. Riveted joints of the simple types dealt with in this chapter are to be found in the connections between members of steel roof trusses and girders (for examples see Figs. 218 and 273). Such joints may be grouped into three classes: (*a*) lapped joints, see Fig. 216; (*b*) butt joints with single cover plates, see Fig. 217; and (*c*) butt joints with double cover plates, see Figs. 217 and 218.

Rivets and rivet holes. In structural work the sizes of rivets commonly used are "$\frac{3}{4}$ in.", "$\frac{7}{8}$ in." and "1 in."; these dimensions refer to the nominal diameter (d) of the rivet.

The bar stock from which rivets are made, however, is frequently $\frac{1}{32}$ in. less than the stated diameter (d) of the rivet, e.g. bar for making "$\frac{3}{4}$ in." rivets would be $\frac{23}{32}$ in. in diameter or $(d - \frac{1}{32})$ in.

The holes made in the plates to receive the rivets may be formed either by drilling or punching, the former method being preferable, since it causes the least damage to the material surrounding the hole. These holes are made slightly larger than the diameter of the rivet bar, for ease of fixing and also because the rivet expands on being heated. This allowance does not usually exceed $\frac{1}{16}$ in., the hole then being $(d + \frac{1}{32})$ in. in diameter.

The red-hot rivet, after being placed in position, is "closed", that is a "head" is formed on it either by hammering or by means of hydraulic or other pressure. This pressure tends to make the rivet expand and fill up the whole of the rivet hole, which, as we have seen, may actually exceed the nominal diameter (d) of the rivet. In practice, however, it is usual to base the calculations for the strength of the joint upon the nominal diameter (d) of the rivet.

As the rivet cools it contracts in length and thus presses the plates of the joint tightly together. The friction thereby set up between the plates increases the strength and rigidity of the joint; in British practice no allowance for friction is made in calculating the strength of the joint.

158. Stresses in simple riveted connections. A very simple riveted connection, in which the two portions of a bar are lapped for a short distance and secured by a single rivet, is shown in Fig. 215. Such a joint is known as a "lapped joint". It will be convenient to consider the stresses which would be set up in the various parts of such a simple joint when subjected to a pull P.

(a) Shear stress on the rivet. It will be clear that in such a case the rivet will be subjected to shear stress—described as "single

shear"—on the plane C–C, where the two portions of the bar are in contact, see Fig. 215 (a). If s be the *shear stress* set up in the rivet on this plane, then evidently

$$\begin{aligned} P &= \text{shear stress} \times \text{area of rivet} \\ &= s \times \frac{\pi d^2}{4}. \qquad \text{......(i)} \end{aligned}$$

(b) **Tensile stress in the bar.** If, as shown, the *breadth* of the bar be b, this will be reduced at the section D–D (taken through the centre of the rivet) to a width of $(b - d)$, see Fig. 215 (b). If then the *thickness* of the bar be th, the area of the cross section of the bar at this section will be $(b - d)\,th$. Obviously the greatest tensile stress will occur at this section and, if t stands for the *tensile stress*, we have

$$\begin{aligned} P &= \text{tensile stress} \times \text{reduced area of bar} \\ &= t\,(b - d)\,th. \qquad \text{......(ii)} \end{aligned}$$

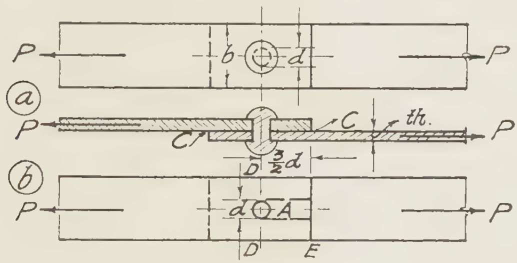


Fig. 215. Simple riveted lapped joint.

(c) **Bearing stress on the rivet.** The pull P also tends to crush the rivet on one side, as at A in Fig. 215 (b), or, correspondingly, to crush the curved side of the hole upon which the rivet *bears*, the stress so induced is therefore commonly referred to as a "bearing stress"; it is in fact merely a compressive stress acting over a small area, or a "localised compressive stress".

The actual stresses set up over the curved surface of the rivet (or hole) do not lend themselves to simple analysis. An approximation which is commonly adopted in calculations, and which is supported by experimental results, is to assume that the bearing stress is distributed uniformly over the "projected area" of the rivet (or the hole), that is, over an area equal to the thickness of the plate multiplied by the diameter of the rivet, or ($th \times d$). Hence, if c_b be the *bearing stress*, we have

$$\begin{aligned} P &= \text{bearing stress} \times \text{projected area of rivet} \\ &= c_b\,(d \times th). \qquad \text{......(iii)} \end{aligned}$$

(d) Other stresses. In addition to inducing the stresses in the joint which have already been indicated, the pull P will also tend to "shear" or force out the portion of the plate at A in front of the rivet; see Fig. 215 (b). Failure in this way is not likely to occur, however, if there is an adequate amount of material between the rivet and the end of the bar. The following rule, which is also justified by practical considerations, is in common use and will be adopted in later calculations. *The centre of a rivet should not be less than one and one-half times the diameter of the rivet from the end of the plate*: or DE in Fig. 215 (b) should not be less than $\frac{3}{2}d$.

(e) Working stresses for riveted joints. The following working stresses, which correspond to those adopted in practice, will be used; see also Table VII.

Shear stress for rivets $= s = 5\frac{1}{2}$ tons per sq. in.
Tensile stress in bars $= t = 7\frac{1}{2}$ tons per sq. in.
Bearing stress $= c_b = 11$ tons per sq. in.
Rivets in "double shear" (see later notes) may be taken to be $1\frac{3}{4}$ times the strength of rivets in single shear.

159. Example 1. *A tie bar which is $2\frac{1}{2}$ ins. wide and $\frac{3}{8}$ in. thick is to be connected by a simple lapped joint and a single rivet; see Fig.* 216 (a).

Adopting a working shear stress of $5\frac{1}{2}$ tons per sq. in., find the maximum pull P which the joint will sustain. Find also the bearing stress and the tensile stress for this load.

(a) From expression (i) in para. 158 we have

$$P = \text{safe shear stress} \times \text{area of rivet}$$
$$= \frac{11}{2} \times \frac{\pi}{4} \times \left(\frac{3}{4}\right)^2 = 2{\cdot}43 \text{ tons},$$

which is the strength of a $\frac{3}{4}$ in. rivet in single shear.

(b) From expression (iii) in para. 158 we have

$$P = c_b\,(d \times th),$$

or $$2{\cdot}43 = c_b\,(\tfrac{3}{4} \times \tfrac{3}{8}),$$

whence $$c_b = \text{bearing stress on rivet}$$
$$= 2{\cdot}43 \times \tfrac{4}{3} \times \tfrac{8}{3}$$
$$= 8{\cdot}64 \text{ tons per sq. in.},$$

which is well within the safe working stress in bearing.

(c) From expression (ii) in para. 158 we have

$$P = t\,(b - d)\,th,$$

whence we have

$$\text{Tensile stress} = t = \frac{P}{(b-d)\,th}$$
$$= \frac{2{\cdot}43}{(2\frac{1}{2} - \frac{3}{4})\,\frac{3}{8}} = 3{\cdot}7 \text{ tons per sq. in.}$$

From this last result it will be seen that in this case, even when the shear stress reaches the full working value, the tensile stress is only

about half that which may be allowed. To get a more effective joint we may use two rivets, and alter the width if necessary so as to give equal strength in shear and tension.

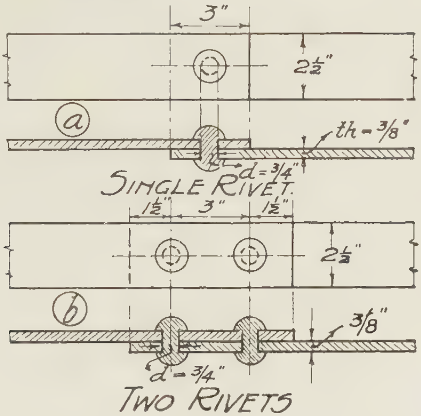


Fig. 216. Simple lapped joint.

Example 2. *Using two rivets as shown in Fig. 216 (b), find the value of P and the width of the bar if the joint is to be of equal strength in tension and shear.*

Proceeding as in the last example it is clear that, since each rivet is capable of carrying a load of 2·43 tons in single shear, two rivets will carry a load of (2 × 2·43) or 4·86 tons, which decides P in this case. Using a tensile working stress of $7\frac{1}{2}$ tons per sq. in. we have

$$4{\cdot}86 = 7{\cdot}5\,(b - \tfrac{3}{4})\,\tfrac{3}{8},$$

whence $$(b - \tfrac{3}{4}) = 1{\cdot}75 \text{ ins. approx.},$$

or $$b = \text{total width}$$
$$= 2\tfrac{1}{2} \text{ ins.}$$

The other dimensions of the joint may be arranged as shown in Fig. 216 (*b*). The bearing stress will be the same as in the preceding example, since two rivets are being used to carry twice the load.

160. Butt joints. The use of lapped joints is usually restricted to small and simple structures. In larger structures, where the loads to be carried may be considerable, the two members to be connected usually butt against each other and are secured by a plate or a pair of plates—known as "cover plates"—across the joint, the whole being connected by rivets; see Fig. 217. These cover plates are made substantially thicker than the main members, so that, provided the distribution of the rivets is satisfactory, the strength of the cover plates need not usually be considered separately in designing the joint. (In important joints the cover plates would be separately designed.) If a single cover plate is used, this may be made at least $\frac{5}{4}$ the thickness of the main

members. If two cover plates are used, each may be made equal to at least $\frac{5}{8}$ the thickness of the bars; see Fig. 217.

Where a lapped joint, or a butt joint having a single cover plate, is subjected to a pull P, the parts of the joint tend to bend as shown in the sketch in Fig. 217, thus subjecting the joint to bending stresses in addition to those already specified. Deductions are not usually made from the strength of these joints on account of this defect, but, wherever possible, double cover plates should be used, since the tendency to bend is thus effectively removed and a more satisfactory joint obtained. Where double cover plates are used it will be seen that the rivets are in "double shear", the rivets tending to shear along planes C–C and D–D when the joint is subjected to a tensile force P. To allow for possible inaccuracies in the alignment of the holes *the strength of a rivet in double shear is usually taken to be not twice, but only* $1\frac{3}{4}$ *times the strength of a rivet in single shear.*

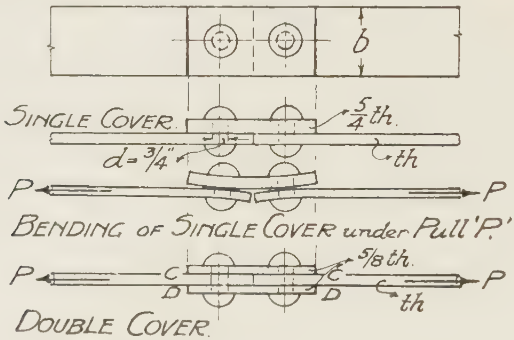


Fig. 217. Simple butt and cover joints.

Example. *If, as in the previous examples, the joint shown in Fig.* 217 *be formed in a tie bar* $2\frac{1}{2}$ *ins. wide and* $\frac{3}{8}$ *in. thick, ascertain the maximum safe load which the joint will carry if double cover-plates are used.*

The safe strength of a $\frac{3}{4}$ in. rivet in double shear will be $1\frac{3}{4} \times 2{\cdot}43$ (see previous Example) or 4·25 tons.

At this load the tensile strength of the bar will be sufficient, see last worked example.

The bearing stress at this load will be given by

$$c_b = \frac{P}{d \times th} = \frac{4{\cdot}25}{\frac{3}{4} \times \frac{3}{8}}$$

$$= 15 \text{ tons per sq. in. (approx.).}$$

This stress is clearly too high (due to the comparatively thin bar which is used), so that the strength of the joint will be limited by the bearing stress. Using a working stress in bearing of 11 tons, we have

$$P = 11\left(\tfrac{3}{4} \times \tfrac{3}{8}\right)$$

$$= 3 \text{ tons (approx.).}$$

The joint would be improved by using two rivets or a thicker bar.

161. A large tension joint. In the case of larger riveted connections, such as that shown in Fig. 218 between a large tension bar and a gusset plate in a framed girder, a considerable number of rivets will have to be used. These may be arranged in the form of "chain riveting" or "zig-zag riveting"; see Fig. 218. As will be seen from the worked example given below, the choice of this

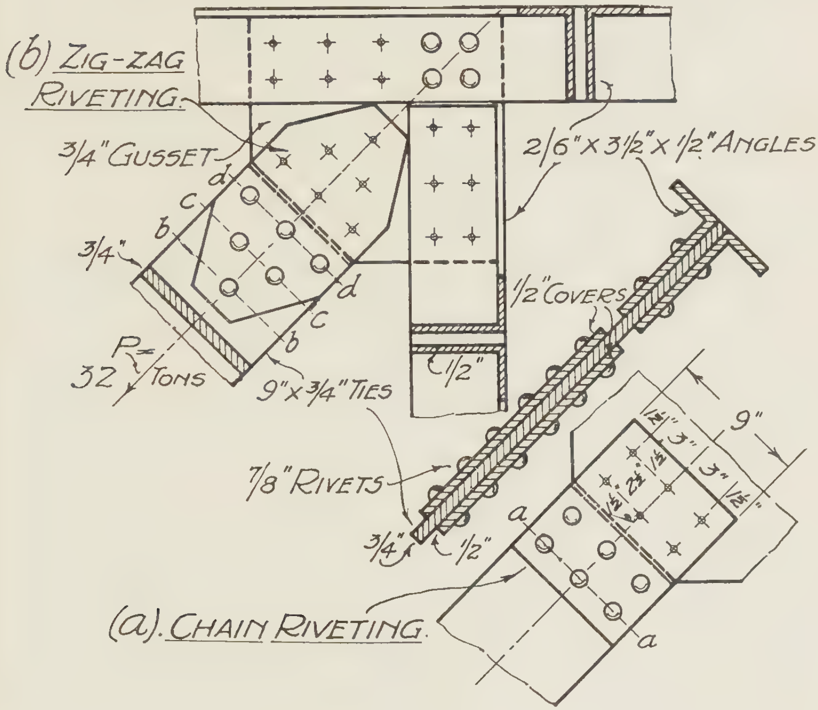


Fig. 218. Joint in large tie bar.

arrangement of the rivets has an important effect upon the "efficiency" of the joint. *The efficiency of a riveted joint is given by the ratio between the minimum strength of the joint and the strength of the solid plate.* Expressed as a *percentage* this is

$$\text{Efficiency } \% = \frac{\text{minimum strength of joint}}{\text{strength of solid plate}} \times 100. \quad \ldots\ldots\text{(iv)}$$

Example. *The tie bar shown in Fig. 218 is to be made from $\frac{3}{4}$ in. plate and attached by means of a butt joint with double cover plates to a $\frac{3}{4}$ in. gusset plate. The maximum load on the bar is 32 tons. Using $\frac{7}{8}$ in. rivets, design a suitable connection and find the efficiency of the joint.*

The thickness of the cover plates should be ($\frac{3}{4} \times \frac{5}{8}$) or $\frac{15}{32}$ in., say $\frac{1}{2}$ in. plate.

The strength of a single $\frac{7}{8}$ in. rivet in double shear

$$= 1\tfrac{3}{4} \times 5\tfrac{1}{2} \times \pi/4 \times (\tfrac{7}{8})^2$$
$$= 5{\cdot}8 \text{ tons (approx.)}.$$

Hence from shear strength the number of rivets required is

$$(32/5{\cdot}8) = 5{\cdot}5,$$

say six rivets, when the minimum strength in double shear will be 34·8 tons.

In bearing, the strength of a single rivet would be

$$(11 \times \tfrac{7}{8} \times \tfrac{3}{4}) = 7\tfrac{1}{4} \text{ tons (approx.)}.$$

As this is greater than for a single rivet in double shear, six rivets will give ample strength in bearing.

(a) **Chain riveting.** It will be assumed in the first place that the rivets are to be arranged as chain riveting, in two rows of three rivets, the spacing of the rows being as shown in Fig. 218 (a).

Pitch. *The distance* (p) *between any two rivets in the same row is known as the "pitch".* In the present case this can be conveniently made 3 ins.; see Fig. 218 (a).

Let us consider the strength of the tie bar on the section a–a, which cuts through three rivet holes. If b_1 be the width of solid metal on this section, then the total width of the bar will be $[b_1 + (3 \times \frac{7}{8})]$. Evidently, if the safe tensile stress is $7\frac{1}{2}$ tons per sq. in., we have

$$32 \text{ tons} = 7\tfrac{1}{2}\,(b_1 \times th) = 7\tfrac{1}{2} \times b_1 \times \tfrac{3}{4},$$

that is

$$b_1 = 32 \times \tfrac{2}{15} \times \tfrac{4}{3}$$
$$= 5{\cdot}7 \text{ ins. nearly.}$$

The total width of the bar would therefore be ($5{\cdot}7 + 3 \times \frac{7}{8}$) or 8·325 ins., say 9 ins.

Efficiency. In this case the minimum strength is shown to be 34·8 tons (in shear). The strength of the solid bar is ($7\frac{1}{2} \times 9 \times \frac{3}{4}$) or 50·625 tons. Hence

$$\text{Efficiency } (\%) = \frac{34{\cdot}8}{50{\cdot}625} \times 100 = 68{\cdot}74\ \%.$$

The above figure for the efficiency is not very high; this is in part due to the fact that on section a–a the plate is weakened by three rivet holes. We shall now show that the arrangement shown in Fig. 218 (b), where we use a zig-zag arrangement, gives better results.

(b) **Zig-zag riveting.** Assuming that the general dimensions remain as before, we may find the minimum strength of the joint by assuming failure in several possible ways.

At section b–b the tie bar is weakened by one rivet hole. The safe tensile strength at this section is therefore $7\frac{1}{2}\,(9 - 0{\cdot}875)\,\frac{3}{4}$ or 45·7 tons.

If failure took place at section c–c, then one rivet would have to be sheared (double shear) and the tie bar torn across at a section weakened by two rivet holes. The safe strength at section c–c is therefore

$$5{\cdot}8 + 7\tfrac{1}{2}\,(9 - 2 \times \tfrac{7}{8}) \times \tfrac{3}{4}$$
$$= 5{\cdot}8 + 7{\cdot}5 \times 7{\cdot}25 \times \tfrac{3}{4}$$
$$= 46{\cdot}7 \text{ tons.}$$

If failure took place at section *d–d*, then *either* (i) three rivets would have to be sheared and the tie bar torn across where it is weakened by three holes, *or* (ii) the two cover plates would have to be torn across at this section.

(i) The safe strength on this assumption will be

$$(3 \times 5{\cdot}8) + 7\tfrac{1}{2}\,(9 - 3 \times \tfrac{7}{8}) \times \tfrac{3}{4}$$
$$= 53{\cdot}26 \text{ tons (approx.).}$$

(ii) The safe strength on this assumption will be

$$7\tfrac{1}{2}\,(9 - 3 \times \tfrac{7}{8})\,\tfrac{1}{2} \times 2$$
$$= 47{\cdot}8 \text{ tons (approx.).}$$

Comparing the various results it will be clear that the first one gives the minimum strength of the joint. The efficiency of the joint (%)

$$= \frac{45{\cdot}7}{50{\cdot}625} \times 100 = 90{\cdot}27\ \%.$$

While the efficiency of the joint is now very satisfactory, it will be seen that (with zig-zag riveting) the bar and joint are both stronger than is required. The bar and joint might therefore be re-designed, using either a thinner or a narrower bar (but see the notes in the next paragraph on the general design of riveted joints).

162. The design of riveted joints. A plan which is commonly adopted in the design of all but the more complex types of riveted joints, consists of *making the joint equally strong in shear and tension, then testing to see whether the bearing stress is within safe limits.* (The first portion of the Example in para. 161 was dealt with in this way.)

An alternative method, which has the additional advantage of relating the thickness of the plate to the diameter of the rivets to be used, is to *select a plate of thickness* (th) *so that the bearing strength and the shear strength* (*single or double as required*) *of a single rivet shall be equal.*

Thus, in the case of *single shear*, and using the working stresses given above, we have

total shear resistance of one rivet

= total bearing strength of one rivet,

or, from expressions (i) and (iii) in para. 158,

$$s \times \pi d^2/4 = c_b\,(d \times th),$$

whence, if $s = 5\frac{1}{2}$ tons per sq. in. and $c_b = 11$ tons per sq. in., then

$$th = \text{thickness of plate} = 0{\cdot}394d, \text{ or say } 0{\cdot}4d.$$

In the case of *double shear* this becomes

$$th = \text{thickness of plate} = 0{\cdot}7d, \text{ nearly.} \qquad \ldots\ldots\text{(v)}$$

I. Then, for the given working stresses, if the thickness (*th*) exceeds either of the above ratios, the joint should be designed for equal strength in shear and tension.

If the thickness (th) is less than either of the above ratios, the joint should be designed for equal strength in bearing and tension.

Example. *Using $\frac{3}{4}$ in. rivets, design a tie bar joint with double cover plates to carry a load of* 15 *tons.*

As the rivets are in double shear, make the thickness of the plate

$$= 0{\cdot}7d = 0{\cdot}7 \times 0{\cdot}75 = 0{\cdot}525, \text{ or say } \tfrac{5}{8} \text{ in.}$$

Then strength of a single $\frac{3}{4}$ in. rivet in double shear

$$= 1{\cdot}75 \times 5\tfrac{1}{2} \times \pi/4\,(0{\cdot}75)^2 = 4{\cdot}25 \text{ tons.}$$

Then four rivets will be ample (giving 17 tons). If two rivets are used in the breadth (b) of the bar, then the strength of the joint in tension $= 15 = 7\frac{1}{2}\,(b - 2 \times 0{\cdot}75)$, from which $b = 3{\cdot}5$ ins. This is hardly sufficient to accommodate the rivets and a bar 4 ins. wide would be used.

163. Pitch of rivets in Compound Girders. In certain circumstances it may be convenient to strengthen an ordinary rolled steel girder by means of plates riveted to its top and bottom flanges; see Fig. 219. Such a girder is known as a **Compound Girder.** Calculations are necessary in order to obtain the pitch of the rivets. The rivets are usually placed zig-zag so as to weaken the section at the most by two rivet holes; see Fig. 219.

The discussion of the distribution of shear stresses in solid beams (see Chap. xv) should make it clear that in the present case the chief work of the rivets will be to carry the horizontal shear forces acting along the planes A–A and B–B. It thus becomes necessary to ascertain the magnitudes of these forces. This can be done most readily in this case by adopting the methods described in para. 137, Chap. xv, and applying statement IV. As that statement indicated, "the total shear force acting on any horizontal layer in a beam, between the two selected vertical sections, will be represented by the 'volume' of the stress-difference diagram above that layer".

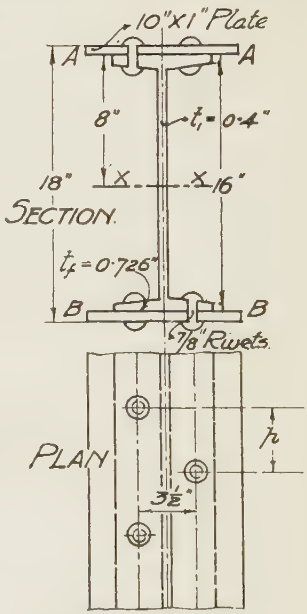


Fig. 219. Compound beam.

In this Example, since the layer A–A is so near to the compressive edge of the beam, we may, as a very close approximation, assume that the stresses acting across the flange plate above A–A at any section are uniform, and equal to the maximum stress at the compression (or tension) edge.

The Example given below will explain the procedure to be followed.

Varying the pitch in the length of the beam. As will be seen from the Example, the pitch is determined for the portion of the beam near the end support, which is usually the position of greatest shear. Theoretically the *pitch* should vary with the shear force, being least near the ends of the beam, where the shear force is high, and greatest over the centre of the beam, where the shear force is small. In practice, however, it is usually much less costly to adopt one (or two) pitches to be used over the whole length of the beam; if two are used the larger one is used over the centre portion of the beam.

Example. *The compound girder shown in Fig.* 219 (*rolled section* 16 *in.* × 6 *in.* × 50 *lbs.*) *is to be used to carry a uniformly distributed load of* 36 *tons over a span of* 20 *ft. Calculate the pitch of the rivets* ($\frac{7}{8}$ *in.*) *and find the mean shear stress in the web and the maximum stresses set up in the flanges.* (*The average thickness of the flanges is* 0·726 *in.*)

Flange stresses. To find these stresses we must first find the value of I for the compound section; see Fig. 219. Referring to Appendix I we obtain the value of I for the rolled section, which is 618 inch units^4. To this must be added the I of each flange plate about X-X. Consider the upper flange, the I for this about X–X will be made up of (i) the I of the plate about its own axis, together with (ii) the area of the plate in section multiplied by the square of the distance from its centroid to X–X. Since the term (i) is very small it may be omitted. Then I for two plates about X–X

$$= (10 \times 1)\,(8{\cdot}5)^2 \times 2 = 1445 \text{ inch units}^4 \text{ (approx.)}.$$

We must next note that the weakest section of the beam contains two rivet holes. For this the deductions to be made from I are: (i) the I of the rivet hole area about its own axis, together with (ii) the magnitude obtained by multiplying the area of the hole by the square of the distance of its centroid (approximately 8 ins.) from X–X. Term (i) is very small and will be neglected. The value of the second term for two rivets is

$$2\,(\tfrac{7}{8} \times 1{\cdot}726)\,8^2 = 193 \text{ inch units}^4 \text{ (approx.)}.$$

Then the value of I for the whole section is given by

$$I = 618 + 1445 - 193 = 1870 \text{ inch units}^4.$$

To find the flange stresses we shall need the value of Z. This we obtain from the expression

$$Z = \frac{I}{y},$$

where y is half the depth (d) of the beam, 9 ins. in this case. Then

$$Z = \tfrac{1870}{9} = 208 \text{ inch units}^3 \text{ (approx.)}.$$

(**Note.** Z could have been found very approximately by means of the expression $Z = Ad$, see para. 132.)

The maximum bending moment (B) will occur at the centre of the beam and be equal to $WL/8$, hence we have

$$B = \frac{WL}{8} = \frac{36 \times 2240 \times 20 \times 12}{8} = 2{,}420{,}000 \text{ lb. ins. (approx.)}.$$

Then from $B = fZ$, where f is the flange stress, we have

$$f = \frac{B}{Z} = \frac{2,420,000}{208}$$
$$= 11,640 \text{ lbs. per sq. in. (nearly).}$$

Maximum force to be carried by flange rivets. The maximum shear force (S) occurs at the points of support and equals ($W/2$) or 18 tons. Then B_d = maximum difference in bending moment in 1 in. of length of beam (see Example, para. 138)

$$= 18 \times 2240 \times 1$$
$$= 40,320 \text{ lb. ins.}$$

Then $\quad f_d$ = maximum stress difference $= \dfrac{B_d}{Z} = \dfrac{40,320}{208}$

$$= 194 \text{ lbs. per sq. in.}$$

Then the maximum shear force to be resisted by the rivets between the flange and the girder *per inch of length*

= (maximum stress difference) × (net area of flange)

$= 194\,(10 - 0{\cdot}875)\,1 = 1770$ lbs. ......(a)

Resistance of rivets. The rivets will be seen to be in single shear. The shear strength of a $\frac{7}{8}$ in. rivet $= 5\frac{1}{2} \times \pi/4 \times (\frac{7}{8})^2 = 3{\cdot}3$ tons.

In bearing, the rivets will be weakest where they pass through the flange of the rolled section, the average thickness of which is only 0·726 in. Hence the strength of one rivet in bearing

$$= 11 \times 0{\cdot}726 \times \tfrac{7}{8} = 7 \text{ tons.}$$

The minimum resistance of one rivet is therefore 3·3 tons. ......(b)

Pitch. Let p be the pitch or distance between the rivets in inches as measured along the length of the beam; see Fig. 219. Then *the total force to be resisted by one rivet in the length* p *will be given by* (*total stress difference per inch*) × (*pitch*), *and this must evidently be equal to the resistance of one rivet.* Hence, using the values obtained in (a) and (b) above, we have

$$1770 \times p = 3{\cdot}3 \times 2240,$$

or

$$p = \text{pitch} = \frac{3{\cdot}3 \times 2240}{1770}$$
$$= 4{\cdot}17 \text{ ins., say a 4 in. pitch.}$$

Since the shear force falls off rapidly in magnitude towards the centre we may use a larger pitch of say 6 ins. over the centre portion of the beam.

Mean stress in web. To calculate the stress in the web we may take its depth as the distance between the flanges of the rolled section, that is (approx.) 14·5 ins. The area of the web is then (14·5 × 0·4) or 5·8 sq. ins. Then the maximum mean shear stress is therefore 18/5·8 or 3·1 tons per sq. in. This is rather high. The web stress should lie between $2\frac{1}{2}$ and 3 tons per sq. in.—the usual working stresses—and the lower value is the better one to use (see Table VII).

164. Plate girders. Where girders are required of larger dimensions than can be conveniently built up from rolled sections, they may be constructed entirely of plates and angles. An example is

shown in Fig. 220. The design of such girders, except in the simplest cases, involve considerations which would carry us beyond the limits of this volume, but, since the problem shows some interesting applications of previous work, we will deal with a simple case in the following worked example.

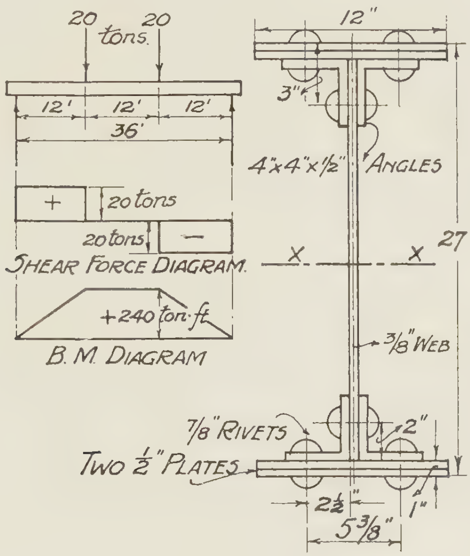


Fig. 220. Plate girder.

Example. *In a large floor one of the main beams, shown in Fig.* 220, *receives from the secondary girders two equally spaced loads of* 20 *tons each ; the span being* 36 *ft. Design a suitable section for a plate girder, limiting the web stress to* 6000 *lbs. per sq. in. and the flange stress to* 16,000 *lbs. per sq. in. It may be assumed that the weight of the girder itself has been allowed for.*

Shear force and bending moment diagrams are given in Fig. 220; it will be seen that the shear force reaches a maximum and remains uniform over the end portions, where it is equal to 20 tons, or 44,800 lbs. The bending moment reaches a maximum value of 240 ton ft. or 6,450,000 lb. ins. at the two load points, and is uniform over the whole of the centre portion.

Depth of beam. It may be shown that, for a given working flange stress, if the deflection is limited to a definite fraction of the span (L), the depth of a beam must bear a definite ratio to the span; thus, if the deflection is not to exceed $\frac{1}{500}$ span, the depth (d) must not be less than $\frac{1}{16}$ span. *In ordinary structural work* (d) *may range between* L/12 *to* L/16. In this case we will make d equal to $\frac{1}{16}$ of the span, or

$$d = \frac{36}{16} = 2 \cdot 25 \text{ ft.} = 27 \text{ ins.}$$

Width of beam. *The width of the flanges* should similarly bear some relation to the span and *may be made between $\frac{1}{30}$ and $\frac{1}{40}$ of the span.* In the present case we will make the flanges 12 ins. wide, which is just $\frac{1}{36}$ of the span.

Web. We may take the depth of the web to be $(27 - 2)$ or 25 ins. If its thickness be *th*, its area will be $(th \times 25)$. The maximum shear force being 44,800 lbs., we have

$$44{,}800 = 6000\,(th \times 25),$$

from which we have

$$\text{Thickness of web} = \frac{44{,}800}{6000 \times 25}$$
$$= 0{\cdot}3 \text{ in.}$$

This is too thin; for practical reasons it would be advisable to use a web not less than $\frac{3}{8}$ in. thick. (For an outside position $\frac{1}{2}$ in. would be better as affording some allowance for probable corrosion.)

Web stiffeners. In any case such a thin web would have to be strengthened against buckling, since its depth is considerable. This could be done by means of vertical angle irons, reaching from one flange to the other, and placed at intervals equal to the depth of the beam and at load points. Such stiffeners might be omitted over the centre portion of the beam and only used between the ends and the load points.

Flanges. Angles are used to connect the flanges to the web. The thickness of these angles should usually be rather more than that of the web. The width will depend upon the proposed width of the flanges. If the flanges are to be very wide, there may be another row of rivets outside the angle irons. We will use $4 \times 4 \times \frac{1}{2}$ in. angles and two rows of $\frac{7}{8}$ in. rivets. In calculating the areas of the flanges the upper portion of the angles may be included; in this case the area to be so added will be $(2 \times 4 \times \frac{1}{2})$ or 4 sq. ins.

To ascertain the necessary area of the flanges we use the expression $B = fZ$, from which, knowing the value of B (6,450,000 lb. ins.) and f (16,000 lbs. per sq. in.), we can find the value of Z. The value of Z may be expressed approximately by $Z = Ad$ (see para. 132), when, knowing d, the depth, we can find A, the area of the flanges. From $B = fZ$ we have $6{,}450{,}000 = 16{,}000Z$, or

$$Z = \frac{6{,}450{,}000}{16{,}000} = 400 \text{ inch units (approx.).}$$

Then from the approximate expression $Z = Ad$, where d, the depth of the beam, is 27 ins., we have

$$A = \text{area of one flange} = \frac{Z}{d}$$
$$= \frac{400}{27} = 14{\cdot}8 \text{ sq. ins.}$$

Allowing 4 sq. ins. for the angles, we have 10·8 sq. ins. left for the plates. If these are made 12 ins. wide and the thickness is 1 in., the area of 12 sq. ins. will make up for the reduction due to one rivet hole.

As it will be desirable to shorten the plates in the manner described below, two $\frac{1}{2}$ inch plates will be used.

Curtailment of flange plates. If the bending moment diagram is redrawn, as in Fig. 221, it will be seen that the values of the bending moments fall off rapidly beyond the two load points. Now it should be

clear from what we have already done that, *if the depth of a beam remains constant, then the areas of the flanges may vary with the bending moment.* For practical reasons it is only possible to do this roughly. What is usually done, therefore, is to cut off a plate about two rivet pitches beyond the point at which it becomes unnecessary. The procedure is as follows; see Fig. 221. The bending moment diagram has its maximum height *ad* divided into as many parts as there are plates—the tops of the angles being included. These divisions are made to bear the same ratios to each other as do the areas of the plates, in this case as 6 : 6 : 4; see Fig. 221 for a geometrical construction.

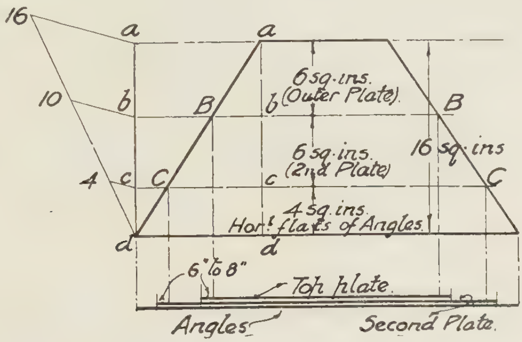


Fig. 221. Curtailment of flange plates.

The points B, B thus obtained give the points beyond which the top plate may be cut. The second plate could be cut at points C, C, but there would be little practical gain in so doing as the points are so near the ends of the beam. In addition, to provide a suitable bearing at the bottom for the beam, and also the necessary protection and finish to the web angles on the top, the inner flange plates would certainly be continued to the ends in such a case.

Joints in the flange plates. If, owing to the length of the beam, it is necessary to butt-joint one or other of the flange plates, a cover plate must be provided. The length of the cover plate must be sufficient to ensure that there is no loss in strength in the flange when considered as a separate riveted joint.

Pitch of rivets. This may be readily found in the manner explained in the Example in para. 163. The following alternative method may, however, be used in this case.

In determining the thickness of the web for this girder we adopted the approximate rule that the web carries the whole of the shear force (S) at any section. If we think of this total shear force, at any section such as AB in Fig. 222, as being made up of a number of vertical shear stresses, we may think of these stresses as being accompanied everywhere throughout the section by equal horizontal and complementary shear stresses; see para. 135 and small figure (a) in Fig. 222. The total effect of these horizontal shear stresses must evidently be communicated to the flanges, through the rivets connecting the web to the flanges.

Consider first the rivets connecting the web and the flange angles, and let these be at a height h apart, and at a horizontal distance apart equal to the pitch p. Then if R be the minimum resistance of one rivet and

S be the shear force acting at any section AB, and if the sections AB and CD each pass through two of these rivets, so that the distance between AB and CD equals p, then, taking moments about one of the rivets at D, we must have, for equilibrium between the horizontal and vertical forces due to R and S,

$$S \times p = R \times h,$$

or
$$p = \text{pitch of rivets} = \frac{Rh}{S}. \qquad \text{......(vi)}$$

In the present case the minimum resistance of one rivet may be in bearing, since such a thin web plate is being used.

Strength of one rivet in bearing $= 11 \times \frac{3}{8} \times \frac{7}{8} = 3{\cdot}61$ tons.

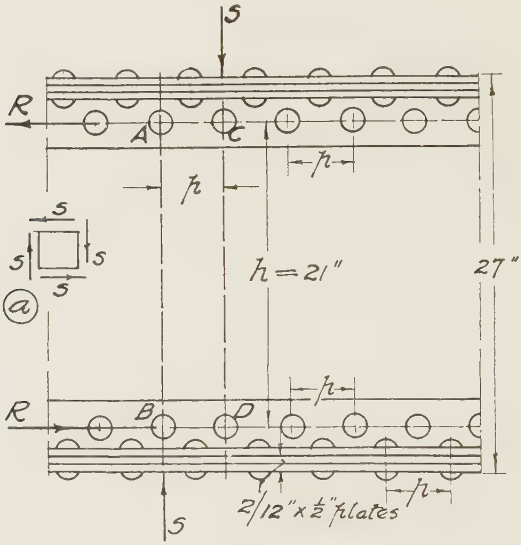


Fig. 222. Pitch of rivets in a built-up beam.

This is clearly less than the strength of a $\frac{7}{8}$ in. rivet in double shear (5·8 tons, see Example, para. 161).

The maximum value of S, the shear force, will be 20 tons at the end supports.

The value of h, the height between the centres of the web rivets,

$$= 27 - 2\,(3) = 21 \text{ ins.};$$

see Figs. 220 and 222.

Then
$$\text{Pitch of rivets} = \frac{Rh}{S} = \frac{3{\cdot}61 \times 21}{20}$$
$$= 3{\cdot}79, \text{ or say } 3\tfrac{1}{2} \text{ ins.}$$

The same pitch may be used in this case for the flange rivets. (These rivets are in single shear and $h = 25''$). A separate calculation should be made to determine the pitch, taking account of shear and bearing. A uniform pitch for the web and flange rivets is desirable, using the smaller pitch resulting from the calculations.

Over the centre portion of the beam the pitch might be increased to 6 ins., but *in no case should the longitudinal distance between two rivets exceed* 16 *times the thickness of the thinnest plate which is to be connected.* In the above case it should be noted that the longitudinal distance between the rivets in the flange plates is equal to $(2p)$, or twice the pitch.

In designing the above section we have used the approximate expression for Z. If desired the value of I and Z may be re-calculated by the more exact methods already described and suitable adjustments made if such are found necessary. The errors will probably, however, be found to be small and on the side of safety.

Connections to web. When it is necessary to connect secondary or other girders to the web, short lengths of angle iron are used and sufficient rivets (or bolts) provided to resist the reactions of the secondary beams, the strength of the rivets (or bolts) being checked both in shear and in bearing.* (See Problems XVIII, 7.)

* The complete design of floors and of a plate girder is dealt with in *Architectural Building Construction*, Vol. II, Part I, by Jaggard and Drury, to which the reader is referred for various practical details not included in the treatment given in this work.

Problems XVIII

(**Note.** The working stresses given in Table VII should be used in solving the following problems.)

1. Using the second method described in para. 162, design a simple lapped riveted joint with a single $\frac{7}{8}$ in. rivet. Give the thickness and width of the bar and state the safe tensile load which it will carry.

2. Design a double cover butt joint for the tie bar of a roof truss to carry a load of 24,000 lbs., using $\frac{7}{8}$ in. rivets. Give the number of rivets on each side of the joint, the thickness and width of the bar, and calculate the efficiency of the joint.

3. Re-design the joint shown in Fig. 218 so that, with zig-zag riveting, the minimum strength of the joint is 32 tons. Use $\frac{7}{8}$ in. rivets but calculate the other dimensions of the joint and its efficiency.

4. A compound girder is to be made up as shown in Fig. 219 from a 12 in. × 8 in. rolled steel joist with two flange plates 10 in. × $\frac{3}{4}$ in. Taking the necessary particulars from the Table given in Appendix I, find the moment of inertia of the compound section (allowing for rivet holes). Find the minimum pitch of the rivets ($\frac{7}{8}$ in. diameter), if the girder is to be used over a span of 16 ft. and loaded with a uniformly distributed load which produces maximum bending stresses of $7\frac{1}{2}$ tons per sq. in. in the beam.

5. Find I and Z for the plate girder shown in Fig. 220, allowing for the holes made by the rivets in the flanges.

6. Calculate suitable details for a plate girder 36 ins. deep and 14 ins. wide, using 6 in. × 6 in. × $\frac{5}{8}$ in. angles and two 14 × $\frac{1}{2}$ in. plates in each flange. What uniformly distributed load would such a beam carry over a span of 45 ft.? What would be a suitable thickness for the web in this case and the minimum pitch of the rivets ($\frac{7}{8}$ in. diameter)?

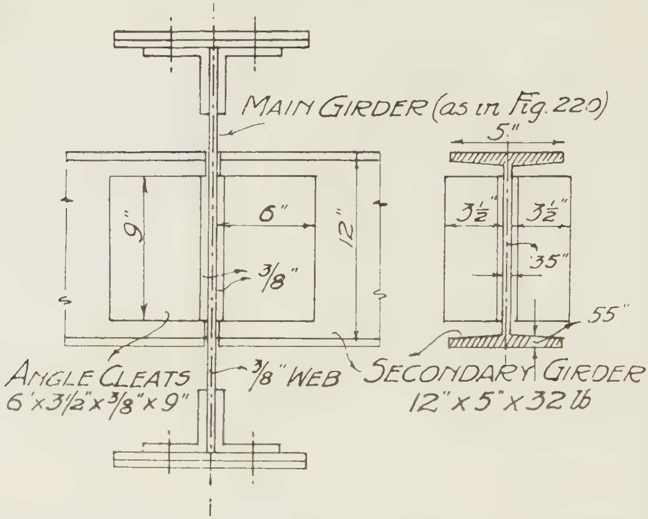


Fig. *A*.

7. The secondary beams attached to the plate girder shown in Fig. 220 consist of 12 in. × 5 in. × 32 lbs. rolled steel sections. These beams are connected to the web of the main girder by angle cleats, of which the details are given in Fig. *A*. If the load transmitted from each girder in this way is 10 tons, find the necessary number of $\frac{3}{4}$ in. rivets or bolts to be used in the web of the secondary and the main beams.

Table VII

Strength and Working Stresses of Steel and Iron

(All stresses in tons (or lbs.) per sq. inch)

Structural (mild) steel

Weight: 490 lbs. per cu. ft.
Modulus of elasticity (E): 13,000 to 14,000 tons. (Use 13,500 tons or 30,000,000 lbs.)
Elastic limit: 17 to 18 tons.
Minimum elongation (on 8 ins.): 20 %

	Ultimate strength	Working stresses
Tension	28 to 32 tons	$7\frac{1}{2}$ tons (or 16,000 lbs.)
Compression	—	$7\frac{1}{2}$ tons ($6\frac{1}{2}$ in columns)
Shear	24 to 26 tons	$5\frac{1}{2}$ tons ($2\frac{1}{2}$ to 3 in webs of beams)

Rivet steel

Ultimate strength: 26 to 28 tons
Elastic limit: 15 to 17 tons
Elongation: 25 %

Working stresses for riveted joints

Tension: $7\frac{1}{2}$ tons. Bearing: 11 tons. Shear: $5\frac{1}{2}$ tons.

Wrought iron

Weight: 480 lbs. per cu. ft.
E: 12,000 to 13,000 tons
Elongation: 20 to 45 %

	Ultimate strength	Working stresses
Tension	20 to 24 tons	5 tons
Compression	16 to 22 tons	5 tons
Shear	15 to 18 tons	4 tons

Cast iron

Weight: 440 to 470 lbs. per cu. ft.
E: 12,000 to 13,000 tons

	Ultimate strength	Working stresses
Tension	7 to 11 tons	$1\frac{1}{2}$ tons
Compression	35 to 50 tons	6 tons
Shear	8 to 13 tons	$1\frac{1}{2}$ tons

CHAPTER XIX

CONCRETE. THE FLITCH BEAM. THE REINFORCED CONCRETE BEAM

165. Concrete. The use of concrete as a constructional material will be sufficiently familiar to the reader to make it unnecessary to give here a detailed statement concerning the materials of which it is composed or of the manner in which it is prepared.* Nor would it be wise to attempt to summarise in a brief paragraph some of the results of the investigations which have been conducted in recent years into the many factors affecting the strength as well as the other properties of concrete. For such information the reader should refer to the several excellent textbooks which have been published on the subject of Reinforced Concrete.

In this volume our immediate concern is with the strength and elasticity of reinforced concrete, as being the main factors affecting its use as a structural material. For this purpose we may briefly describe a cement concrete as consisting of cement, which in this country is usually prepared to the requirements of the British Standard Specification for Portland Cement (see summary in Table VIII), together with sand and coarse aggregate. Such a concrete is characterised by considerable compressive strength but slight tensile strength, a fact which leads readily to a convenient classification of the material according to its use (*a*) as Mass concrete, or (*b*) as Concrete used in preparing "reinforced concrete".

(a) Mass concrete. This form of concrete is largely used in foundations, rough walling, etc., and may consist of varying proportions of cement, sand and aggregate, according to the nature of the materials used and the purposes for which it is to be employed. Unless specially prepared such concrete would not as a rule be subjected to a greater working load than 12 tons per sq. ft. (see Table X). Since it is practically incapable of resisting tensile stresses, mass concrete must be considered structurally as forming blocks of material, as in masonry construction, its treatment is therefore deferred to Chaps. XXI and XXII, which deal with Masonry.

(b) Concrete for use in "reinforced concrete". By reason of long-continued use and control, concrete used in the preparation of reinforced concrete has become a material of considerable uniformity and reliability. It follows, however, that its preparation

* But see Jaggard and Drury's *Architectural Building Construction*, Vol. II, Part I.

must be carried out with skill and care, hence both its constituents and the manner of its preparation are now largely determined by regulation. These regulations lay down what are in effect the minimum requirements to which the material must conform. Some allowance is usually made for concrete which may show higher strength values than those specified, but, in order to simplify somewhat the treatment of the subject of reinforced concrete in this volume, it is proposed to limit our consideration to a concrete which satisfies what is the most commonly adopted specification for such material. Such a "normal" concrete is expected to be made up of 1 part Portland cement, 2 parts of sand and 4 parts of aggregate, and is therefore commonly referred to as a "1 : 2 : 4" concrete.

To test the suitability of such concrete, 6 in. test cubes (or cylindrical test pieces) are made up and crushed in a testing machine at ages of 1 month and 4 months, and are expected to show a minimum ultimate crushing strength of 1600 lbs. per sq. in., and 2400 lbs. per sq. in., respectively, at the above ages. The working compressive strength of such concrete would be taken at 600 lbs. per sq. in., or a quarter of the ultimate strength at 4 months. The value of E would be about 2,000,000 lbs. per sq. in. (but see para. 168). Other strength characteristics of concrete are referred to later in this and in the succeeding chapter, and also in Chaps. XXIII and XXIV, to which reference may be made.

Experiment. The following results were obtained from a test carried out on a 1 : 2 : 4 concrete at an age of 3 months. The specimens measured 13 ins. × 4 ins. × 4 ins. The strains were measured by a Goodman's extensometer on a gauge length of 8 ins.

Crushing load (average of 3 specimens), 18 tons.

$$\text{Ultimate strength} = \frac{18 \times 2240}{16} = 2500 \text{ lbs. per sq. in. (approx.).}$$

Modulus of elasticity. The following figures are taken from the graph shown in Fig. 225, which gives the mean values for the strains measured as the loads were being increased, and again as the loads were being decreased. This test was made after the specimen had been subjected to a load of 3 tons, put on and removed twice.

For an increase of stress of 490 lbs., the increase in strain measured 0·002 in.

$$E = \frac{\text{stress}}{\text{strain}} = \frac{490 \times 8}{0{\cdot}002}$$
$$= 1{,}960{,}000 \text{ lbs. per sq. in.}$$

166. The flitch beam—the reinforced timber beam. An early form of beam which depended for its strength upon the combined action of two materials was the "flitch" or "sandwich" beam, in which a timber beam was strengthened by the addition of one or more

plates of wrought iron or steel; see Fig. 223. This form of beam is now almost obsolete in new work, but since, as will appear presently, an investigation into the factors underlying its design requires a consideration of the same basic principles as are utilised to build up the theory of reinforced concrete construction, it has been decided to include it in this volume.

It is usual to assume that the timber and steel parts of a flitch beam are fastened together with sufficient bolts to ensure that the parts move together when the beam deflects—the adequacy of the bolts for this purpose being checked in a manner similar to that described in the preceding Chap. (XVIII) for built-up beams. On this

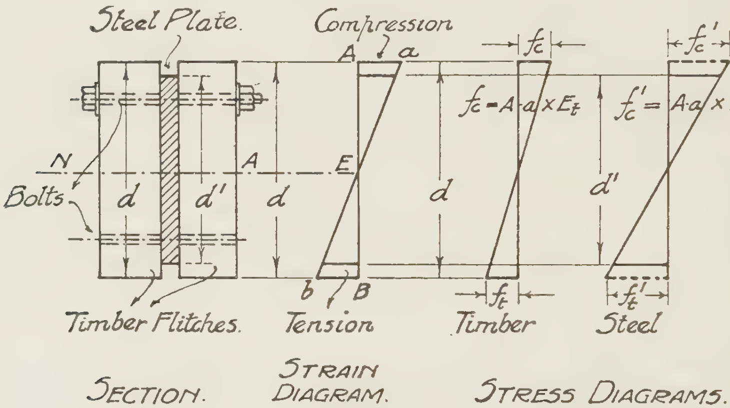


Fig. 223. Flitch beam with steel plate.

basis it will be obvious that we may adopt all the assumptions laid down in para. 116 (Theory of Simple Bending), together with the additional condition that the modulus of elasticity (E) will have two values, viz. E_t (1,500,000 lbs. per sq. in. may be used here) for the timber, and E_s (usually 30,000,000 lbs. per sq. in.) for the steel.

By assumption IV, para. 116, sections which are plane before bending remain plane after bending; hence, if *unit strain* be represented on a diagram as shown in Fig. 223, aEb will be a straight line and Aa will be the maximum compressive strain, while bB will be the maximum tensile strain. In addition the strain will be the same in each material and will vary in each layer according to its distance from the neutral axis. (**Note.** In practice the steel plate does not extend quite to the full depth of the beam, see Example below, but for the present it will be convenient to assume that it does so.)

It follows that, so long as the stresses in the materials are kept within their respective elastic limit stresses, a diagram of stress can be drawn for each material as shown in Fig. 223. In the case of the timber if f_c be the maximum compressive stress, then we have

$$f_c = \text{unit strain} \times \text{modulus} = Aa \times E_t,$$

from which we have

$$Aa = \text{maximum compressive unit strain} = \frac{f_c}{E_t}. \quad \ldots\ldots(a)$$

Similarly, using the diagram for stress in the steel, we have

$$f_c' = \text{maximum compressive stress in steel}$$
$$= Aa \times E_s,$$

or

$$Aa = \text{maximum compressive strain in steel}$$
$$= \frac{f_c'}{E_s}. \quad \ldots\ldots(b)$$

Combining the expressions (a) and (b) we have

$$\frac{f_c}{E_t} = \frac{f_c'}{E_s},$$

whence

$$\frac{f_c'}{f_c} = \frac{\text{compressive stress in steel}}{\text{compressive stress in timber}} = \frac{E_s}{E_t}. \quad \ldots\ldots(c)$$

The ratio E_s/E_t is known as the **Modular Ratio**, and is usually represented by m. In the present case, for the values given above,

$$m = \frac{E_s}{E_t} = \frac{30{,}000{,}000}{1{,}500{,}000} = 20.$$

Then from (c) we have

$$\frac{f_c'}{f_c} = m = 20,$$

or

$$f_c' = 20 f_c,$$

i.e. *the stress in the steel is* 20 (*or* m) *times the stress in the timber*; this statement is true for fibres in each material which are at the same distance from the N.A. of the beam.

Equivalent sections. It follows from the above conclusion that we would have a beam of the same strength, i.e. giving the same maximum stress for the same load, if we were to replace the steel plate by an additional timber beam of m times the thickness of the steel plate, the *stresses* in this "equivalent" beam being the same as in the timber portion of the original beam. This equivalent section furnishes us with the readiest means of dealing with the design of these beams.

Example. *A flitch beam is to be made up of two* 11 *ins. by* 3 *ins. timber beams together with a* $\frac{1}{2}$ *in. steel plate. The safe working stresses may be taken to be* 1000 *lbs. per sq. in. in the timber and* 16,000 *lbs. per sq. in. in the steel,* m *having the value of* 20. (a) *Find for these values the depth of the steel plate so that the maximum working stresses in each material are reached simultaneously, and* (b) *with this width of steel plate what uniformly distributed safe load would the beam carry over a span of* 12 *ft.?*

(*a*) Let the plate be the same depth as the timber slabs; then, when the stress in the timber reached 1000 lbs. per sq. in., the stress in the steel would be m times this, or 20,000 lbs. per sq. in. Since this latter stress will vary as the distance $d'/2$ from the neutral axis, where d' is the depth of the steel plate and d the depth of the timber portion, see Fig. 224, then evidently

$$\frac{d'/2}{d/2} \text{ must be made to equal } \frac{16{,}000}{20{,}000},$$

or

$$\frac{d'}{d} = \frac{d'}{11} = \frac{16}{20},$$

that is, $d' = 8{\cdot}8$ ins., say 9 ins. (nearest market size).

(*b*) The equivalent section will then be as shown in Fig. 224, where the width of the middle portion is ($m \times \frac{1}{2}$) or 10 ins.

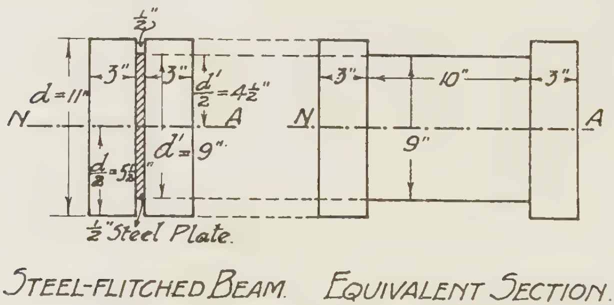


Fig. 224. Flitch beam with steel plate.

Then the value of the "equivalent moment of inertia", I_e, for this equivalent section will be obtained from

$$\begin{aligned} I_e &= I \text{ for timber portion} + I \text{ for "equivalent" portion} \\ &= \frac{2 \times 3 \times 11^3}{12} + \frac{10 \times 9^3}{12} \\ &= 1273 \text{ inch units}^4 \text{ (approx.)}. \end{aligned}$$

Then from $B = fI/y$, we have

$$\begin{aligned} B = \text{bending moment} &= \frac{f \times I_e}{y} = \frac{1000 \times 1273}{5{\cdot}5} \\ &= 231{,}500 \text{ lb. ins. (approx.)}. \end{aligned}$$

Also, for a uniformly distributed load, we have $B = WL/8$; whence, substituting the known values and transposing, we have

$$W = \text{total load} = \frac{231{,}500 \times 8}{12 \times 12} = 12{,}860 \text{ lbs.}$$

167. The reinforced concrete beam. If a plain block of concrete be tested as a beam it will be found to fail, under quite small loads, by tension on the under surface of the beam. Concrete, though strong in compression, is relatively weak in tension, and it is to "reinforce" the concrete in this respect that reinforcing bars of steel are used which, in a beam, are placed near the lower surface in order to take the tensile forces to which the beam will be

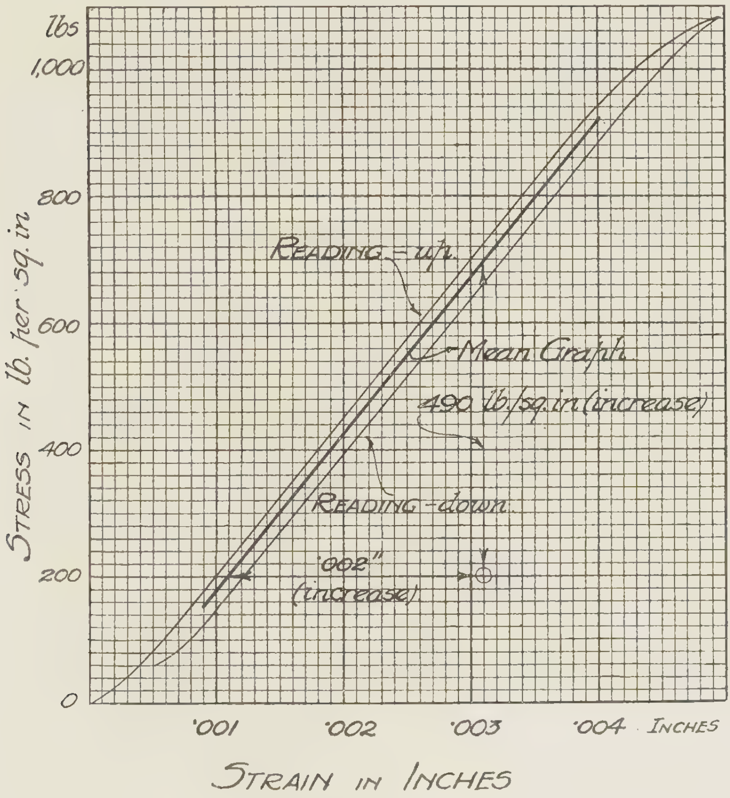


Fig. 225. Stress-strain graph for 1 : 2 : 4 concrete.

subjected as a result of loading. Such beams furnish us with another example of the combined use of two materials in a structure. The theory underlying the construction of such beams is, as we shall see, very similar to that which we used in the case of flitched beams and is, for the simple cases with which we shall deal in this volume, not difficult of comprehension. (For some introductory

experimental work see para. 185, "Tests on reinforced concrete beams".)

In Fig. 226 is shown a typical section of a reinforced concrete beam. As in the case of the flitched beam we are able to adopt the usual beam assumptions (see para. 116, "Theory of simple bending"), to which, however, two important additions are necessary.

Additional Assumption I. That within the working stresses of the concrete in compression, stress and strain are directly proportional to each other, the modulus of elasticity (E_c) having a constant value.

That this assumption is only approximately true for concrete is evident as soon as we attempt to obtain values experimentally from which to find E. When the specimen is being loaded for the first time, some amount of permanent set will be shown even at small loads, due probably to the material "settling down" to its work. It is therefore usually necessary to stress such a concrete to about 400 or 500 lbs. per sq. in. several times, as was done in the present instance, before attempting to obtain satisfactory stress-strain values. The graph shown in Fig. 225 is in fact only approximately straight up to a stress of about 850 lbs., after which it departs markedly from the straight line.

The assumption errs somewhat on the side of safety, it simplifies the calculations which are involved and, in practice, it is found to give satisfactory results.

Additional Assumption II. That the resistance of the concrete to tension may be neglected, the whole of the tensile forces being carried by the reinforcement.

The reasons underlying the adoption of this assumption will be made clear as we proceed.

168. Normal working stresses and constants in reinforced concrete. The following values are those in common use in this country. Other values would of course give different results from those worked out below, the methods of calculation being the same, but it is not our intention to deal with such variations in this volume.

For a concrete which consists of 1 part of cement, 2 parts of sand and 4 parts of aggregate (and which is known as a "1 : 2 : 4" concrete) it is usually assumed that:

> E_c, the modulus of elasticity, is 2,000,000 lbs. per sq. in.
> The safe compressive stress (c) = 600 lbs. per sq. in. (which is about one quarter the ultimate strength of such concrete in compression).

For the quality of mild steel which is generally used the value of the modulus E_s is taken to be 30,000,000 lbs. per sq. in. The safe tensile stress in the steel (t) = 16,000 lbs. per sq. in. (which is about half the elastic limit stress).

For the above values the **Modular Ratio** (m) is seen to have a value of

$$m = \frac{E_s}{E_c} = \frac{30{,}000{,}000}{2{,}000{,}000} = 15.$$

(See also Table VIII.)

169. The equivalent section. Referring again to Fig. 226 and using the usual assumption, that sections which were plane before bending remain plane after bending, a diagram of strain $AaEbB$ can be drawn in which the line aEb is a straight line. If the calculations of the strength of such a beam were based on this diagram, in which the strain of the steel at Gg is shown to be

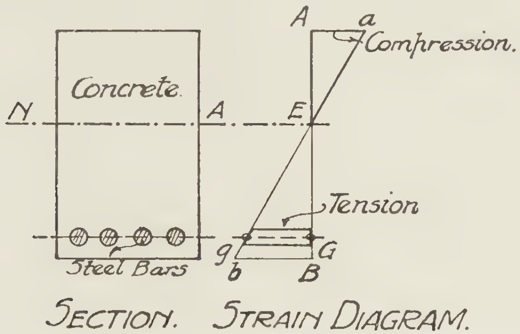


Fig. 226. Reinforced concrete beam.

equal to that in the concrete at the same level, then the stress in the steel would be m times the stress in the concrete at an equal distance from N–A. But concrete, as has already been mentioned, is very weak in tension, the safe tensile stress being generally put as low as 60 lbs. per sq. in. On this basis the stress in the steel would only be equal to (60 × 15) or 900 lbs. per sq. in., which is of course absurdly low.

This then is the reason why, in the theory which we are now about to discuss, the strength of the concrete in tension is entirely disregarded and the steel is assumed to take the whole of the tensile force; in this way the stress in the steel may be assumed to reach the full value of 16,000 lbs. per sq. in.

Let us now consider how these assumptions affect the active section of the beam. If the strength of the concrete in tension be ignored, then it follows that we need not consider the part played

by it below the neutral axis. What we may call the "effective section" of such a beam will then consist of (*a*) the "compressive area" above the neutral axis, and (*b*) the area of the steel bars below the neutral axis, which are of course in tension; see Fig. 227.

Some of the terms used in this discussion may be defined here:

A_t is the area of the tensile reinforcement.

b is the breadth of the beam.

d is the "effective depth" of the beam, that is the depth from the "compression edge" to the centre of the reinforcement.

n is the depth to the neutral axis from the compression edge.

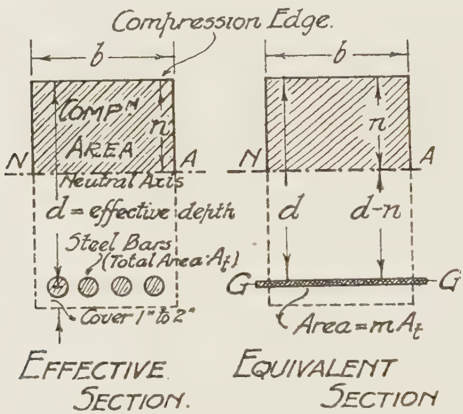


Fig. 227. Equivalent section.

Below the reinforcement a thickness of from 1 to 2 ins. of concrete, known as the "cover", is usually added, but, except when ascertaining the weight of the beam, this dimension need not appear in the calculations. Such concrete is added to provide additional grip to the reinforcement and also as a protection against corrosion and fire.

From the "effective section" just outlined we may develop an "equivalent section", on the same principles as applied in the case of the flitch beam; see Fig. 227. It would consist of the rectangular "compression" area, $b \times n$, above the neutral axis, and the "tension" area, equal to m times the area of the reinforcement, situated at a distance $(d - n)$ below the neutral axis; see Fig. 227.

By means of this equivalent section we might deduce all the general expressions (i) to (viii) given below; and the method is particularly useful in certain cases, of which examples are given later in paras. 177 (A) and 179 (A). We propose, however, to

proceed from this point along the lines more usually adopted, which have perhaps a special claim to our consideration, in that they give a somewhat clearer picture of what takes place in the strained reinforced concrete beam.

170. General expressions in the ordinary theory of reinforced beams. The theory which is outlined below, and which is based on the assumptions stated above, is known as the **Straight Line No-tension Theory**: "Straight Line" because the stress diagram for the concrete in compression will be a triangle, and "No-tension" because the effect of the concrete in tension is ignored.

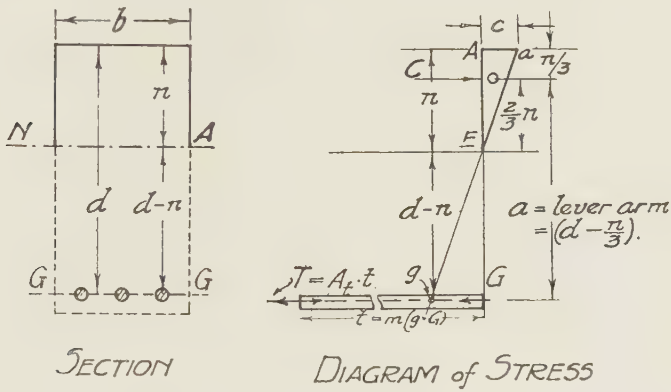


Fig. 228. Straight line no-tension theory.

In Fig. 228 the section has been re-drawn, together with a diagram of stress obtained in the following manner. The compressive stresses in the concrete above the neutral axis will vary uniformly from a maximum value c at the compression edge to zero at the neutral axis, and the diagram of stress can thus be represented by the triangle AaE in which Aa represents the stress c to some convenient scale.

Now if C be the total compressive force acting over the area $(b \times n)$ we have

$$C = \text{total compressive force} = \tfrac{1}{2}bnc. \qquad \text{......(i)}$$

Since the reinforcing bars generally occupy only a small portion of the depth of the beam, we assume that the stress in them is uniform. Then if T be the total tensile force and (t) the stress acting in the bars, we have

$$T = \text{total tensile force} = A_t.t. \qquad \text{......(ii)}$$

Again, since in simple bending the total compressive force is equal to the total tensile force, we have from (i) and (ii)

$$C = T,$$

or $$\tfrac{1}{2}bnc = A_t . t. \qquad \ldots\ldots(\text{iii})$$

Now C will act on a line passing through the centroid of the stress triangle AaE, that is $\tfrac{2}{3}n$ above the neutral axis or $\tfrac{1}{3}n$ from the compression edge, this point being sometimes referred to as the "centre of compression". It will thus be seen from the diagram that the distance between the two parallel forces C and T will be equal to $(d - n/3)$, this is referred to as the "lever arm" (a). Obviously if we wish to ascertain R, the moment of resistance of this beam section, we must multiply C, or T, by the lever arm; hence

Moment of resistance of the section $= R$,

and $$R = C \times \text{lever arm} = \frac{1}{2} bnc \left(d - \frac{n}{3}\right), \qquad \ldots\ldots(\text{iv})$$

or $$R = T \times \text{lever arm} = A_t . t \left(d - \frac{n}{3}\right). \qquad \ldots\ldots(\text{v})$$

Let us next proceed to find an expression giving the relations between the stresses c and t. If, for the moment, we assume that concrete is a material capable of resisting tensile stresses satisfactorily, then the stress in the concrete at the level G–G, see Fig. 228, would be given on the stress diagram by the distance Gg, where g is the point in which the line aE produced cuts the horizontal line through G, and Gg is measured on the force scale selected for c. But, the strain being the same in both materials at this level, the stress in the steel would be m times that in the concrete, or

$$t = \text{stress in steel} = m\,(Gg),$$

whence $$Gg = t/m.$$

But, by similar triangles,

$$\frac{Aa}{AE} = \frac{c}{n} = \frac{Gg}{EG} = \frac{t/m}{d - n},$$

from which we have

$$\frac{c}{n} = \frac{t}{m\,(d - n)},$$

or $$\frac{mc}{t} = \frac{n}{d - n}. \qquad \ldots\ldots(\text{vi})$$

From (vi) we have further

$$\frac{t}{mc} = \frac{d - n}{n} = \frac{d}{n} - \frac{n}{n} = \frac{d}{n} - 1,$$

from which

$$\frac{d}{n} = 1 + \frac{t}{mc},$$

or

$$\frac{n}{d} = \frac{1}{1 + \frac{t}{mc}}. \qquad \text{......(vii)}$$

As will be seen from the worked example given below, the expressions (i) to (vii) are sufficient to enable us to ascertain the strength of a given reinforced concrete beam at a section, when the values of c, t and m are known.

Ratio of reinforcement. A symbol "r" is used in this and succeeding examples. It represents the ratio between the area (A_t) of the tensile reinforcement and the "effective area" ($b \times d$) of the beam and is known as the "ratio of reinforcement", so that

$$r = \frac{A_t}{bd}.$$

If this ratio is to be expressed as a percentage, then percentage reinforcement equals $100r$.

171. The "Economic Ratio" of Reinforcement.

Example 1. *Find the value of* n *and of* r, *the ratio of reinforcement, for a beam in which the value of* m *is* 15, *of* c *is* 600 *and of* t *is* 16,000.

From (vii) we have

$$\frac{n}{d} = \frac{1}{1 + \frac{t}{mc}},$$

so that, substituting the given values,

$$\frac{n}{d} = \frac{1}{1 + \frac{16,000}{15 \times 600}} = 0{\cdot}36,$$

or

$$n = 0{\cdot}36d. \qquad \text{......(a)}$$

Next from (iii) we have

$$\tfrac{1}{2}bnc = A_t.t,$$

when, substituting $0{\cdot}36d$ for n, we have

$$\tfrac{1}{2}bc \times 0{\cdot}36d = A_t.t,$$

whence we have

$$\frac{A_t}{bd} = \frac{0{\cdot}18c}{t} = \frac{0{\cdot}18 \times 600}{16,000} = 0{\cdot}00675,$$

or

$$\text{Ratio of reinforcement} = r = \frac{A_t}{bd} = 0{\cdot}00675. \qquad \text{......(b)}$$

From (iv) we have

$$R = \text{moment of resistance} = \frac{1}{2}\,bnc\left(d - \frac{n}{3}\right)$$

$$= \frac{1}{2} \times 0{\cdot}36d \times 600 \times b\left(d - \frac{0{\cdot}36d}{3}\right) = 95bd^2. \qquad \text{......(c)}$$

It should be noted that the numerical relations (a), (b) and (c) found in the above example only obtain when the value of m is 15, c is 600 and t is 16,000. Under these conditions, if the ratio of reinforcement is 0·00675, then the stresses in the concrete and in the steel reach their maximum working values together when the beam is fully loaded. The constants are independent of the actual dimensions of the beam, so that if they are memorised, the calculation of beam sections is greatly simplified. When the ratio (r) has the value of 0·00675 it is known as the "economic ratio of reinforcement". (Other values for m, c and t would of course give other constants.)

Example 2. *A reinforced concrete lintel is required to carry a uniformly distributed load of 6500 lbs. (which may be taken to include the weight of the lintel), over a span of 12 ft. If the total depth is to be 12 ins., find a suitable breadth and the dimensions of the reinforcement. Take the values of* c *as 600 lbs. per sq. in., of* t *as 16,000 lbs. per sq. in. and of* m *as 15.*

Allowing 2 ins. for the cover of the reinforcing bars, the effective depth (d) will be 10 ins.

The maximum bending moment = R = moment of resistance

$$= \frac{WL}{8} = \frac{6500 \times 12 \times 12}{8}. \quad \text{......(a)}$$

If $r = 0{\cdot}00675$, then R also $= 95bd^2 = 9500b$. ......(b)

Combining (a) and (b) we have

$$b = \frac{6500 \times 144}{8 \times 9500} = 12{\cdot}32, \text{ say } 12\tfrac{1}{2} \text{ ins.}$$

Then Area of reinforcement $= A_t = r \times b \times d$

$$= 0{\cdot}00675 \times 12{\cdot}5 \times 10$$

$$= 0{\cdot}84 \text{ sq. in.}$$

If we use three $\frac{5}{8}$ in. round bars (see Table IX), they provide $(3 \times 0{\cdot}307)$ or an area of 0·921 sq. in. which is satisfactory though a little in excess.

172. Beam sections where only *c* or *t* is known. In practice it is not always possible to arrange the reinforcement so that r has the value worked out above, so that it is necessary to work out some more general expression which we can use when c or t are known but not both; the expression we shall obtain will give the ratio n/d in terms of r.

From (vi) in para. 170 we have

$$\frac{mc}{t} = \frac{n}{d-n},$$

or

$$\frac{c}{t} = \frac{n}{m(d-n)}. \quad \text{......(a)}$$

Also, from (iii) in the same paragraph, we have

$$\tfrac{1}{2}bnc = A_t . t,$$

but $A_t = r\,(bd)$, therefore

$$\tfrac{1}{2}bnc = rb\,dt,$$

or
$$\frac{c}{t} = \frac{2rd}{n}. \qquad \text{......(b)}$$

Equating the two values for c/t in (a) and (b), we have

$$\frac{n}{m\,(d-n)} = \frac{2rd}{n},$$

whence, taking d out of the bracket and re-writing,

$$\frac{n}{dm\,(1-n/d)} = 2r\,(d/n).$$

If now we substitute n_1 for n/d, we have

$$\frac{n_1}{m\,(1-n_1)} = \frac{2r}{n_1}.$$

Multiplying across we have

$$\begin{aligned} n_1^2 &= 2mr\,(1-n_1) \\ &= 2mr - 2mn_1r, \end{aligned}$$

or
$$n_1^2 + 2mn_1r = 2mr.$$

Make the left-hand side of the expression into a square by adding m^2r^2, then

$$n_1^2 + 2mn_1r + m^2r^2 = 2mr + m^2r^2,$$

whence, taking the square root of each side,

$$n_1 + mr = \pm\sqrt{2mr + m^2r^2}.$$

The negative value is not possible in the cases we shall have to deal with; hence this expression may be re-written as

$$n_1 = \frac{n}{d} = \sqrt{2mr + m^2r^2} - mr. \qquad \text{......(viii)}$$

By further changes we may reduce expression (viii) to

$$n_1 = mr\left(\sqrt{1 + \frac{2}{mr}} - 1\right). \qquad \text{......(viii. a)}$$

From either of these expressions we may find the value of the ratio n/d, which gives the position of the neutral axis, when the values of d, m and r are known.

Example. *Find the value of the ratio* n/d *for a beam section in which the value of* r *is* 0·01 (*that is* 1 % *of reinforcement*) *and that of* m *is* 15.

Substituting these values in the expression (viii), we have

$$\begin{aligned} \frac{n}{d} &= \sqrt{2mr + m^2r^2} - mr \\ &= \sqrt{0{\cdot}3 + 0{\cdot}0225} - 0{\cdot}15 = 0{\cdot}416. \end{aligned}$$

173. Diagrams to facilitate beam calculations –Position of the neutral axis. The use of the above expressions is greatly facilitated if they are given in the form of graphs. By means of the expression (viii) we may find a number of related values which can be shown on a diagram as in Fig. 229.

Thus, when

$r = 0{\cdot}0025$	$n/d = 0{\cdot}2415$
$0{\cdot}005$	$0{\cdot}319$
$0{\cdot}00675$	**$0{\cdot}36$**
$0{\cdot}008$	$0{\cdot}384$
$0{\cdot}01$	$0{\cdot}416$
$0{\cdot}02$	$0{\cdot}531$, etc.

Having from these and similar values drawn a graph as shown, Fig. 229, giving the position of the neutral axis for various values of r, we can then show the positions of *the centre of compression* by dividing, in each case, the distance n by 3. Finally the value of a, *the lever arm*, is then determinable in each case by reading off the distance from the bottom of the graph to the line showing the centre of compression; see Fig. 229. The use of this graph will be made clear in the following examples.

An example is given to show how the stresses in the steel and the concrete are affected when r has a value less than 0·00675.

Example. *If in a reinforced concrete beam the reinforcement is* 0·5 %, *find the value of* c *if the steel is stressed to* 16,000 *lbs. per sq. in.*

In this case $r = 0{\cdot}5/100 = 0{\cdot}005$. Referring to this value in Fig. 229, n/d is given as 0·319, or $n = 0{\cdot}319d$.

Taking expression (vi) in para. 170, we have

$$\frac{mc}{t} = \frac{n}{d-n}$$

and, inserting the known values, we have

$$\frac{15c}{16{,}000} = \frac{0{\cdot}319d}{d - 0{\cdot}319d} = \frac{0{\cdot}319}{1 - 0{\cdot}319},$$

from which $c = 500$ lbs. per sq. in. (approx.).

This is lower than the value which may be allowed (600), from which it would appear that, when r is less than 0·00675, the concrete is not stressed up to its full value. Similarly, as will appear in later examples, when r exceeds 0·00675, though the concrete be stressed to the full value of 600, the stress in the steel will be less than 16,000. In effect we may say that *for reinforcement below the economic ratio the strength of the beam is determined by the strength of the steel, the concrete not being fully stressed; while for ratios above* 0·00675 *the strength will be determined by the strength of the concrete, the steel in this case bearing a stress which is less than* 16,000.

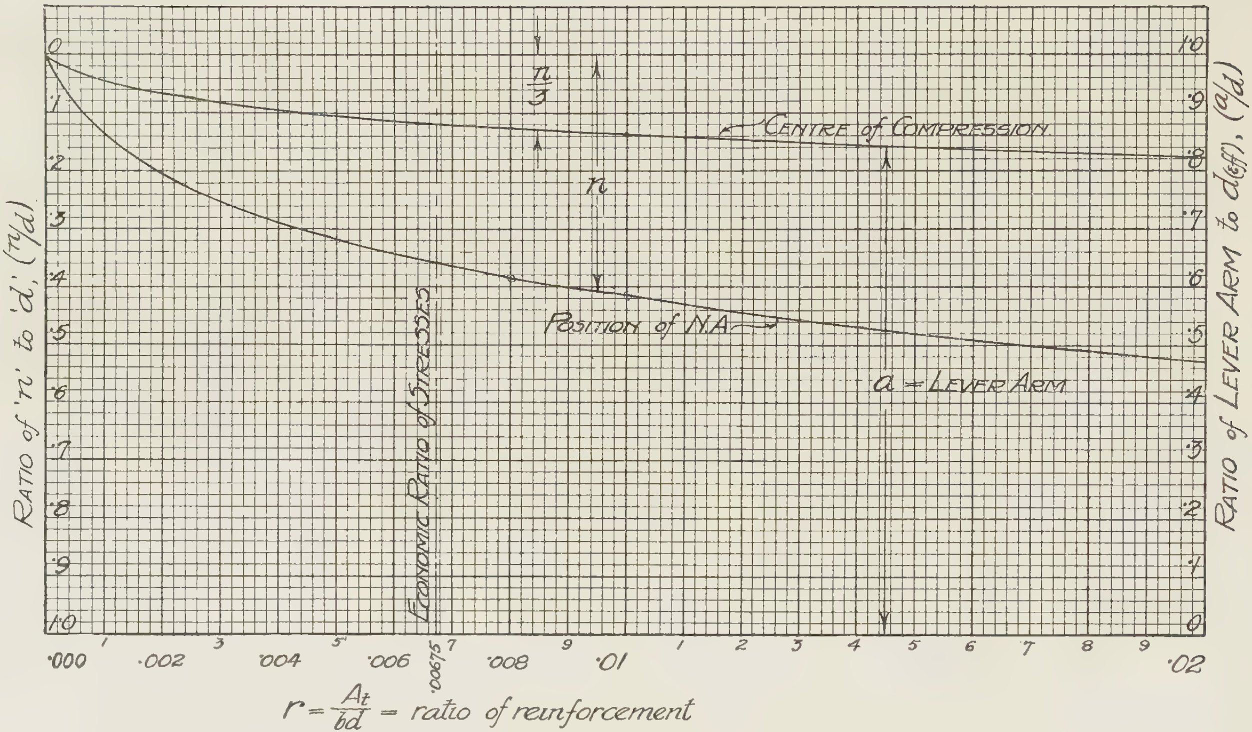


Fig. 229. Position of neutral axis (N.A.) when $m = 15$.

174. Diagram showing how the strengths of beam sections vary with *r*. The above statement may be represented on a diagram wherein the magnitude of R, the moment of resistance, is expressed in the form $R = k\,(bd^2)$, where k is a constant which varies with r. For convenience in use the values of k are plotted, where

$$k = \frac{R}{bd^2}.$$

The following examples will show how the value of k is found.

Example 1. *Find* R *in terms of* b *and* d *when* r *is* 0·005.

This is the case just treated in which we saw that, though the stress in the concrete was less than 600, the stress in the steel did reach the full value of 16,000. We therefore use this value to find R, inserting it in the expression (v) from para. 170, where

$$R = A_t \,.\, t\,(d - n/3).$$

Now we know that $n = 0{\cdot}319d$ when $r = 0{\cdot}005$. Also

$$A_t = r\,(bd) = 0{\cdot}005\,(bd).$$

Substituting these values in the above expression, we have

$$R = 0{\cdot}005bd \times 16{,}000\,(d - 0{\cdot}319d/3) = 71{\cdot}5bd^2,$$

from which

$$k = \frac{R}{bd^2} = 71{\cdot}5.$$

Example 2. *Find* R *in terms of* b *and* d *when* r *is* 0·01.

In this case, since r is greater than the economic ratio, the concrete will be fully stressed (the reader may make a calculation, as in the example in para. 173, to satisfy himself that this is so); hence $c = 600$ is the value we will use in finding R. This we do by means of expression (iv) from para. 170, from which we have $R = \frac{1}{2}bnc\,(d - n/3)$. In this case, since $r = 0{\cdot}01$, we have from Fig. 229 that $n = 0{\cdot}416d$. Inserting this and other known values in the expression for R, we have

$$R = \frac{1}{2} \times b \times 0{\cdot}416d \times 600\left(d - \frac{0{\cdot}416d}{3}\right) = 107bd^2,$$

or

$$k = \frac{R}{bd^2} = 107.$$

These and other values obtained in the same manner enable a graph to be set out as in Fig. 230. Thus, when

$r = 0{\cdot}0025$	$R = 35{\cdot}7bd^2$	$k = 35{\cdot}7$
$0{\cdot}005$	$71{\cdot}5bd^2$	$71{\cdot}5$
0·00675	**$95bd^2$**	**95**
$0{\cdot}01$	$107bd^2$	107
$0{\cdot}02$	$131bd^2$	131, etc.

The dotted extensions indicate that the curve is really made up of two curves, which intersect on the ordinate through $r = 0{\cdot}00675$. The first portion depends upon t, the stress in the steel, while the second portion depends upon c, the stress in the concrete.

It is of interest to note that the strength of a beam section, which varies with k if b and d remain the same, increases with relative slowness as r is increased beyond the economic ratio.

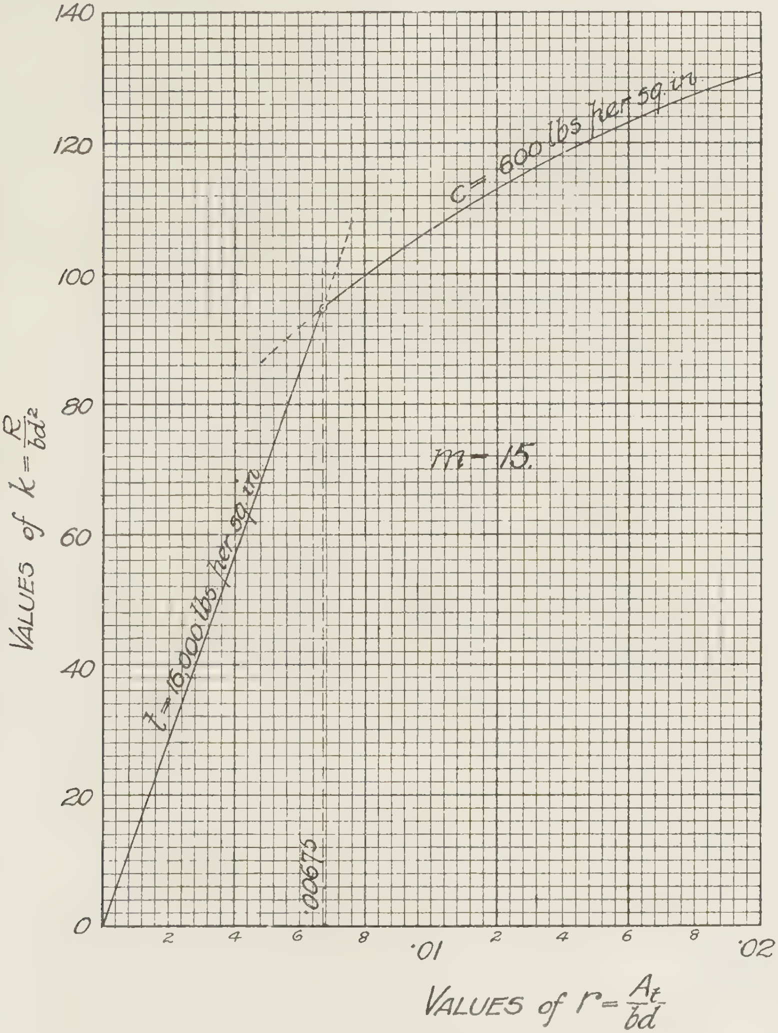


Fig. 230. Values of R/bd^2 for various values of r.

The method of using this curve will be made clear from the following examples. *It should be noted that these calculations really*

depend upon the use of the expressions (i) *to* (viii), *the graphs merely being introduced to facilitate the work.*

Example 3. *A beam* 24 *ins. deep and* 10 *ins. wide is reinforced* 2 *ins. from the lower edge by five* $\frac{3}{4}$ *in. round bars (area of one* $\frac{3}{4}$ *in. bar is* 0·442 *sq. in.). Find the safe distributed load which it will carry over a span of* 20 *ft., making due allowance for the weight of the beam.*

The weight of the beam may be calculated at 150 *lbs. per cu. ft.* Hence

$$\text{Weight of beam} = 20 \times 2 \times \tfrac{10}{12} \times 150$$
$$= 5000 \text{ lbs.}$$

In this case

$$d = 24 - 2 = 22 \text{ ins.}$$
$$A_t = 5 \times 0{\cdot}442 = 2{\cdot}21 \text{ sq. ins.}$$
$$r = \frac{A_t}{bd} = \frac{2{\cdot}21}{22 \times 10} = 0{\cdot}01 \text{ (approx.).}$$

For this value of r we have

$$R = 107bd^2 \text{ (from Fig. 230)}$$
$$= 107 \times 10 \times 22 \times 22$$
$$= 517{,}880 \text{ lb. ins.}$$

Now, for a distributed load,

$$B = \frac{WL}{8} = 517{,}880 \text{ lb. ins.; but } B = R,$$

therefore
$$W = \frac{517{,}880 \times 8}{20 \times 12} = 17{,}260 \text{ lbs.}$$

Subtracting the weight of the beam from this, we have

$$\text{Safe load on beam} = 17{,}260 - 5000 = 12{,}260 \text{ lbs.}$$

Example 4. *A reinforced concrete beam is required to carry a uniformly distributed load of* 8 *tons (including weight of beam) over a span of* 12 *ft. If the total depth has to be limited to* 18 *ins., find a suitable section and give the dimensions of the reinforcement.*

In this case $d = 18 - 2 = 16$ ins.

The required $R = B = \dfrac{WL}{8} = \dfrac{8 \times 2240 \times 144}{8}$

$$= 322{,}500 \text{ lb. ins.}$$

One or two trials may now have to be made to find a value of r which will give a reasonable value of b. Adopting $r = 0{\cdot}007$ we have $k = 97$ from Fig. 230. From this value we have

$$R = kbd^2 = 97b \times 16 \times 16,$$

or
$$(97 \times 16 \times 16)\, b = 322{,}500 \text{ lb. ins.,}$$

whence
$$b = \frac{322{,}500}{97 \times 16 \times 16}$$
$$= 13 \text{ ins. (approx.).}$$

Then

$$A = \text{area of reinforcement} = bdr = 0{\cdot}007 \times 13 \times 16$$
$$= 1{\cdot}46 \text{ sq. ins.}$$

If we use six $\frac{9}{16}$ in. round bars (see Table IX), the total area will be $(6 \times 0{\cdot}248)$, or 1·488 sq. ins.

Table VIII

Working stresses in Reinforced Concrete

Portland cement*

Weight: 90 lbs. per cu. ft.
Fineness test: not more than 10 % on a 180 × 180 sieve.
Tensile test:
(*a*) *Neat cement:* not less than 600 lbs. per sq. in. at 7 days.
(*b*) *Cement and sand* (*mortar*)—1 cement to 3 of standard sand (by weight): not less than 325 lbs. at 7 days.

Breaking strength at 28 days (minimum)

$$= \text{breaking strength at 7 days} + \frac{10{,}000}{\text{breaking strength at 7 days}}.$$

Setting test (except for quick-setting cement): *initial set*, not less than 20 minutes; *final set*, not more than 10 hours.
Soundness test: see B. S. Spec. for Portland cement.

Portland cement concrete (reinforced concrete). "Normal" mixture: 1 part cement, 2 parts sand, 4 parts aggregate.

Weight (aver.): 140 lbs. per cu. ft.; (reinforced): 144 to 150 lbs. per cu. ft.
Modulus of elasticity (E_c): 2,000,000 lbs. per sq. in.
Ultimate strength: 1 month, 1600 lbs. per sq. in.; 4 months, 2400 lbs. per sq. in.
Working stress: (in compression, c), 600 lbs. per sq. in.; (shear), 60 lbs. per sq. in.
Grip stress or adhesion: (with hooks), 100 lbs. per sq. in.; (without hooks), 60 lbs. per sq. in.

Reinforcing steel

Weight: 490 lbs. per cu. ft.
Modulus of elasticity (E_s): 30,000,000 lbs. per sq. in.
Ultimate strength: 28 to 32 tons per sq. in.
Working stress: (in tension, t), 16,000 lbs. per sq. in.; (in compression), mc, where c is the stress in surrounding concrete. ($m = E_s/E_c = 15$.)

* See also current issue of British Standard Specification for Portland Cement.

Problems XIX

(**Note.** In working the following problems use the working stresses given in Table VIII.)

1. To compare the strength of ordinary and "rapid hardening" cement concrete, 6 in. cubes of 1 : 2 : 4 concrete were made and tested at 1 month and 4 months respectively. From the following results calculate the ultimate stresses in each case: Portland cement concrete, at 1 month broke at 36 tons; at 4 months broke at 55 tons; rapid hardening cement concrete, 1 month 60 tons, 4 months 80 tons.

2. Using the working stresses given in para. 166, calculate the safe uniformly distributed load which could be carried by a flitch beam over a span of 15 ft., if the beam consists of two timber flitches 12 ins. by 6 ins. and a steel flitch plate 10 ins. by $\frac{3}{4}$ in.

3. Calculate the total compressive strain in a short plain concrete column 4 ft. high, and having a cross section of 100 sq. ins., $E = 2{,}000{,}000$ lbs. per sq. in. and $c = 500$ lbs. per sq. in. (See also Prob. 6 in Problems XII.)

4. Using the "economic ratio" of steel to concrete, find suitable reinforcement for a beam 10 ins. broad and 14 ins. (effective) depth. Calculate the value of the moment of resistance (R).

5. Find the value of R for the beam in Prob. 4 if it were reinforced with four round rods $\frac{5}{8}$ in. in diameter. What would be the stress in the steel when $c = 600$ lbs. per sq. in.?

6. Find the value of R for the beam in Prob. 4 if it were reinforced with four round rods $\frac{1}{2}$ in. in diameter. What would be the stress in the concrete when $t = 16{,}000$ lbs. per sq. in.?

7. Calculate a suitable depth and reinforcement for a beam to carry a uniformly distributed load—including its own weight—of 500 lbs. per ft. run, over a span of 15 ft., if the width of the beam be fixed at 10 ins. and the economic ratio of reinforcement be used.

8. If the outside dimensions of the beam in the fourth Example, para. 174, are width 14 ins. and total depth 20 ins., calculate suitable reinforcement and give the stresses in the steel and concrete when the beam is carrying the full uniformly distributed load of 8 tons over a span of 12 ft.

Table IX

Circumference, Sectional Area and Weight of Reinforcing Bars

Size of rod, ins.		ROUND			SQUARE		
Fractions	Decimals	Area, sq. ins.	Circum., ins.	Weight, lbs. ft. lin.	Area, sq. ins.	Circum., ins.	Weight, lbs. ft. lin.
$\frac{1}{4}$	0·25	0·049	0·785	0·167	0·063	0·10	0·213
$\frac{5}{16}$	0·3125	0·077	0·982	0·261	0·098	1·25	0·332
$\frac{3}{8}$	0·375	0·110	1·178	0·376	0·141	1·50	0·478
$\frac{7}{16}$	0·4375	0·150	1·374	0·511	0·191	1·75	0·651
$\frac{1}{2}$	0·5	0·196	1·571	0·668	0·250	2·00	0·849
$\frac{9}{16}$	0·5625	0·248	1·767	0·845	0·316	2·25	1·076
$\frac{5}{8}$	0·625	0·307	1·964	1·043	0·391	2·50	1·328
$\frac{11}{16}$	0·6875	0·371	2·160	1·262	0·473	2·75	1·607
$\frac{3}{4}$	0·75	0·442	2·356	1·502	0·562	3·00	1·912
$\frac{13}{16}$	0·8125	0·518	2·552	1·763	0·660	3·25	2·245
$\frac{7}{8}$	0·875	0·601	2·748	2·044	0·766	3·50	2·603
$\frac{15}{16}$	0·9375	0·690	2·945	2·347	0·879	3·75	2·988
1	1·0	0·785	3·142	2·670	1·000	4·00	3·403
$1\frac{1}{8}$	1·125	0·994	3·534	3·380	1·266	4·50	4·303
$1\frac{1}{4}$	1·25	1·225	3·927	4·172	1·563	5·00	5·312
$1\frac{1}{2}$	1·5	1·767	4·712	6·008	2·250	6·00	7·650
$1\frac{3}{4}$	1·75	2·405	5·498	8·178	3·063	7·00	10·41
2	2·0	3·142	6·283	10·68	4·000	8·00	13·60

CHAPTER XX

TEE-BEAMS. DOUBLY REINFORCED BEAMS. SHEAR REINFORCEMENT

175. Tee-beams. If in a rectangular beam, such as that shown in section in Fig. 231, we were to omit the shaded portions indicated below the neutral axis, we would not reduce the strength of the beam. We would, however, greatly reduce the dead-weight and hence the cost of the beam. From its shape such a beam is known as a "T-beam"; it is probably in this form that the concrete beam gives its best service. In concrete floor construction an additional

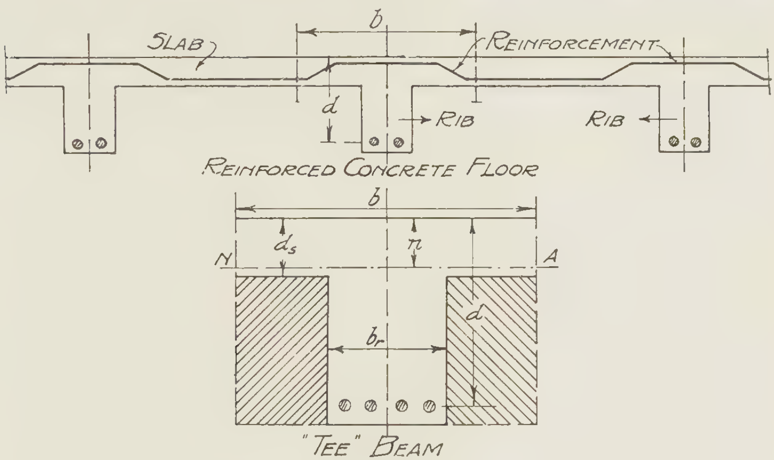


Fig. 231. Tee-beam. Case I. "Thick" slab.

economy can be effected by utilising the floor slab to form the upper flange of the "T"; see (b) in sketch in Fig. 231. The amount of the floor slab which can be so included is not easily settled upon. In practice it is usually fixed by regulation. The following are commonly accepted rules. The dimension b must not exceed:—

(a) One-quarter the span of the beam,
or (b) The distance between two ribs,
or (c) Twelve times the thickness (d_s) of the slab, taking whichever dimension is least.

The same terms will be used as before with the addition of d_s for the *total depth* or thickness of the slab, and b_r for the *breadth* or thickness of the rib. The thickness of the rib will usually be

settled by a consideration of (i) providing adequate cover to the reinforcement, and (ii) providing adequate shear strength; the latter point will be dealt with in para. 184.

There are two cases of T-beams with which we can deal; the first we shall refer to as **Thick Slabs**, meaning by this that the thickness of the slab (d_s) in relation to the depth (d) of the beam is so considerable that the neutral axis lies within, or just at the lower edge of the slab; see Fig. 231. In the second case, which we shall refer to as **Thin Slabs**, the neutral axis will fall outside the slab; see Fig. 232.

176. T-beams—Thick slabs. Beams which fall under this head are much more common in practice than would appear at first sight, since very thin slabs are not favoured, and beams which are *relatively* deep only occur over very large spans.

It should be obvious that where the N.A. falls within the slab the ordinary expressions for rectangular beams will apply in their entirety. In fact, as is pointed out in Faber and Bowie's "Reinforced Concrete Design", only a very small error is introduced if d_s is as small as $0{\cdot}8n$, at any rate it is an error which can be disregarded in view of the arbitrary fixing of (b) the width of the flange. We shall therefore fix this as our limit and *in deciding whether an example falls within Case I we shall ascertain whether* d_s *exceeds* $0{\cdot}8n$.

(**Note.** For the economic ratio it will be seen that, to come within Case I the thickness of the slab (d_s) must not be less than ($0{\cdot}8 \times 0{\cdot}36d$) or $0{\cdot}288d$.)

Example. *A portion of a concrete floor, of which the slab thickness is* 4 *ins., is to be carried on ribs spaced* 8 *ft. apart and freely supported over a span of* 16 *ft. The total load, to be carried as a uniformly distributed load, may be taken to be* 21,000 *lbs. including the weight of the floor. Determine suitable dimensions for the ribs, and the area of the tensile steel.*

According to the rules given above the least dimension (b) for the flange of the "T" is shown by trial to be given by 12 (d_s), or 48 ins.

The value of B = maximum bending moment = moment of resistance

$$= R = \frac{WL}{8} = \frac{21{,}000 \times 16 \times 12}{8}.$$

Adopting a ratio of reinforcement of $0{\cdot}00675$ we have, from Fig. 230, that $k = 95$, or $R = 95bd^2$. Hence we have

$$95bd^2 = \frac{21{,}000 \times 16 \times 12}{8},$$

whence, if $b = 48$ ins., $d = 10{\cdot}5$ ins. (approx.).

From Fig. 229 for $r = 0{\cdot}00675$, we have

$$n = 0{\cdot}36d = 0{\cdot}36 \times 10{\cdot}5$$
$$= 3{\cdot}78 \text{ ins.},$$

which brings N.A. within the thickness of the slab and the T-beam may be treated as a "thick slab" beam.

A_t = area of reinforcement = $bdr = 48 \times 10{\cdot}5 \times 0{\cdot}00675 = 3{\cdot}4$ sq. ins.

This could be provided by four $1\frac{1}{8}$ in. round bars, which would require a rib 10 in. wide if placed in one layer, or 6 in. wide if in two layers.

(Note. Where horizontal layers of reinforcement are used the bars must not be nearer together than 1 in. horizontally and $\frac{1}{2}$ in. vertically. There must be a cover of at least 1 in. at the sides and soffit of the beam. The effective depth of such beams is measured from the compression edge to the centre of the reinforcement.)

177. T-beams—Thin slabs (approximate treatment). In Fig. 232 a T-beam is shown in which the neutral axis comes well below the slab. It should be clear without further explanation that we can in this case draw a diagram of strain and also a diagram of stress

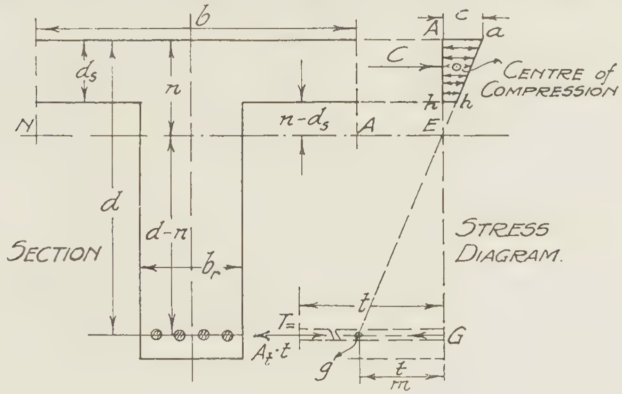


Fig. 232. T-beams—thin slabs.

exactly as for a rectangular beam; see Figs. 226 and 228. In Fig. 232 we have drawn such a diagram of stress, which is exactly the same as that shown in Fig. 228 except that we must consider the compression portion in two parts, viz. that consisting of the figure *Aahh*, which gives the stresses acting in the flange of the T-beam, and the triangle *hhE*, which gives the stresses acting over the area of the rib, between the underside of the slab and the neutral axis. From the diagram a series of general expressions can be obtained for T-beams of the same range and by the same methods as for rectangular beams. The work is, however, too lengthy and complex for inclusion in this volume, and the present treatment will therefore be limited to the explanation of two approximate but reliable methods for dealing with given beam sections. In both these approximate methods, the total force due to the compressive

stresses in the rib is ignored; this can usually be done with safety since both the stresses themselves and the area over which they act are relatively of small dimensions, see Fig. 232.

(A) First approximate method for dealing with T-beams (thin slabs). Equivalent section method. In Fig. 233 is shown the equivalent section for a T-beam, obtained in the manner explained in para. 169, but with the additional condition that the compression area below the slab is ignored.

From this equivalent section we may find the position of the neutral axis, basing our work upon the fact that the first moment

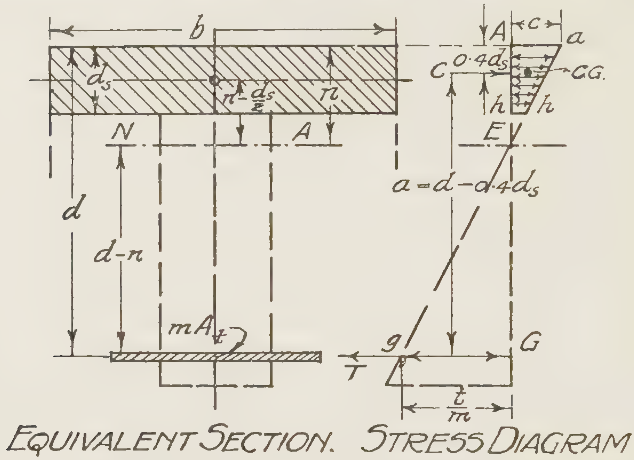


Fig. 233. T-beams—approximate methods.

of the whole "equivalent" section about the neutral axis must be zero. So that, using the dimensions given in Fig. 233, we have

(1st moment of compression area about N.A.)

— (1st moment of tension area about N.A.) = 0,

from which $(bd_s)(n - d_s/2) - (mA_t)(d - n) = 0.$ ......(ix)

From this expression we may find n, the distance to the neutral axis, if b, d, d_s, m and A_t are known.

Having found the position of N.A. we then proceed to find the moment of inertia (I_e) for the equivalent section about N.A. From the work done in para. 131 we see that

I_e = (I for compression area about its own axis)

+ [(area of compression) × $(n - d_s/2)^2$]

+ (I for equivalent tension area about its own axis)

+ (equivalent tension area × $(d - n)^2$).

The third term will be relatively small and can be neglected. Therefore, using the dimensions given in Fig. 233, we have

$$I_e = \frac{bd_s^3}{12} + bd_s\,(n - d_s/2)^2 + mA_t\,(d - n)^2. \qquad \text{......(x)}$$

(Note. The above method can be applied to plain rectangular sections in which case n replaces d_s.)

Having found I_e we then proceed to find R by the methods described in para. 120, in which it was established that

$$B = R = \frac{fI}{y},$$

where f is the stress in the material at any distance y from the neutral axis. Hence, using c for compressive stress and replacing y by n, and considering the stress in the concrete, we have

$$R = \frac{cI_e}{n}. \qquad \text{......(xi)}$$

In considering the stress in the steel we must remember that we are dealing with the "equivalent section" and that the stress to be used is t/m and not t; see the stress diagram in Fig. 233. Hence we have

$$R = \left(\frac{t}{m}\right)\frac{I_e}{d - n}. \qquad \text{......(xii)}$$

Usually the stress in the concrete in T-beams will not reach 600, the expression (xi) need therefore only be used if it is desired to ensure that this value is not exceeded. The following example will make clear the procedure for checking in such a case.

Example. *Find* R *for a* T*-beam, given that* b *is* 36 *ins.*, d *is* 16 *ins.*, d_s *is* 5 *ins. and* $A_t = 4{\cdot}88$ *sq. ins. (four* $1\frac{1}{4}$ *in. round bars). Take* t = 16,000 *lbs. per sq. in.*, c = 600 *lbs. per sq. in. and* m = 15. See Fig. 234.

Inserting the given values in expression (ix), we have

$$bd_s\,(n - d_s/2) - mA_t\,(d - n) = 0,$$

or
$$36 \times 5\,(n - 2{\cdot}5) = 15 \times 4{\cdot}88\,(16 - n),$$

which reduces to $253n = 1621$, or $n = 6{\cdot}42$ ins.

Checking the ratio n/d, we find $n/d = \frac{6{\cdot}42}{16} = 0{\cdot}4$, which is greater than $0{\cdot}36d$. Hence the concrete will reach a stress of 600 lbs. per sq. in. before the steel reaches a stress of 16,000 lbs. per sq. in.

Using this value n in expression (x), we have

$$\begin{aligned} I_e &= \frac{bd_s^3}{12} + bd_s\,(n - d_s/2)^2 + mA_t\,(d - n)^2 \\ &= \frac{36 \times 125}{12} + 36 \times 5\,(6{\cdot}42 - 2{\cdot}5)^2 + 15 \times 4{\cdot}88\,(16 - 6{\cdot}42)^2 \\ &= 375 + 2766 + 6720 = 9861 \text{ inch units}^4. \end{aligned}$$

Then for a stress of 600 lbs. per sq. in. in the concrete we have, using expression (xi),

$$R = \frac{c . I_e}{n} = \frac{600 \times 9861}{6 \cdot 42}$$
$$= 921{,}600 \text{ lb. ins.}$$

To find t for this value of R we use expression (xii), when

$$R = \left(\frac{t}{m}\right) \frac{I_e}{d - n} = \frac{t}{15} \times \frac{9861}{9 \cdot 58},$$

whence t = stress in steel
$= 13{,}430$ lbs. per sq. in.

(B) **Second approximate method for dealing with T-beams (thin slabs). Approximate lever-arm method.** In Fig. 233 let C be the total compressive force acting in the flange. Strictly C will act along a line passing through the centroid of the stress figure $Aahh$. It is not easy to find this point unless the position of the neutral axis, that is point E on this figure, is known. It may be shown, however, that the distance of C from the compression edge will lie between $d_s/3$ (when the neutral axis coincides with the underside of the slab), and $d_s/2$ (when the slab is very thin compared with the depth of the beam). Actually $0 \cdot 4d_s$ *is a very close approximation in the majority of cases* and, since it errs on the side of safety, we may conveniently use this value. The work now proceeds as follows:

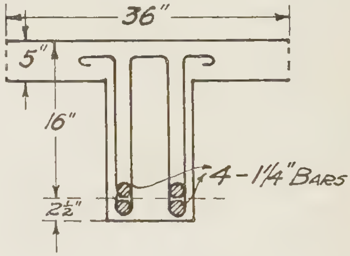


Fig. 234.

If the line of action of C *lies* $0 \cdot 4d_s$ *below the compression edge, then the magnitude of the approximate lever arm will be* $(d - 0 \cdot 4d_s)$.

Hence, if R is given we can find T from the equation

$$R = T \times a \text{ (approx.)} = T\,(d - 0 \cdot 4d_s).$$

We can then find A_t from $T = A_t . t$.

This method is useful where the general dimensions are known, R is given and suitable reinforcement is required.

A rough check on c can be made by remembering that $T = C$ and finding what is the *average* compressive stress over the area of the flange from

$$T = C = (\text{average compressive stress} \times \text{area of flange}).$$

If this average value falls below 500 lbs. per sq. in. a further check is hardly necessary, otherwise it may be checked by method (A).

Example. *Let it be assumed that a beam is required having the same outside dimensions as are indicated in Fig.* 234, *but with a moment of*

resistance (R) *of* 800,000 *lb. ins., and that we desire to find the dimensions of the reinforcement. Assume* t = 16,000 *lbs. per sq. in.*

In this case the approximate lever arm $= a = (d - 0{\cdot}4d_s)$

$$= 16 - (0{\cdot}4 \times 5) = 14 \text{ ins.}$$

Hence $$R = T \times 14 = 800{,}000 \text{ lb. ins.},$$

whence $$T = \frac{800{,}000}{14} = 57{,}000 \text{ lbs.}$$

But $T = A_t.t$, therefore

$$A_t = \frac{T}{t} = \frac{57{,}000}{16{,}000} = 3{\cdot}56 \text{ sq. ins.}$$

This may be provided by six $\frac{7}{8}$ in. round bars.

Since $C = T = 57{,}000$ lbs., the average compressive stress in the flange

$$= \frac{57{,}000}{36 \times 5} = 317 \text{ lbs. per sq. in.},$$

which is well on the safe side.*

178. Floor slabs. The accurate investigation of the strength of reinforced concrete floor slabs is very difficult since, even in the simplest case, they are *supported on four edges* and in the majority of floors they are, in addition, continuous over several spans. As a very rough approximation they may be designed as beams stretching across the shortest span—a width of 10 or 12 ins. being treated in this way as a rectangular beam. Some reinforcement should in such a case be added in a direction at right angles to the first. The necessity for the latter will be seen if we note that, as the slab deflects under load, the concrete will be under tensile stress *in the two principal directions.* The L.C.C. regulations speak of these latter bars as "distribution bars", and require that they should be not less than 0·08 of the cross-sectional area of the reinforcement across the short span.

Floor slabs designed as T-beams—Hollow block floors. In plain reinforced concrete floor slabs the dead weight of concrete is considerable; in order to reduce this weight hollow blocks of burnt clay are sometimes used, the effect, as shown in Fig. 235 (A), being to form, within the thickness of the slab, a series of T-beams. These are designed in the usual way to give the desired strength.

"Filler-joist" floor slabs. Another type of floor slab, and one which is frequently used in steel framed buildings, consists of small steel joists embedded entirely in concrete as shown in Fig. 235 (B). When formed between larger steel beams these slabs can be designed as short freely supported slabs.

* It is essential that the maximum stress in the steel and also in the concrete should be checked in every case, since, unless the dimensions have been predetermined, so as to ensure that $n/d = 0{\cdot}36$, only one material will reach the maximum working stress, and it is important that we should know which material this is.

The strength of these slabs may be calculated *approximately* by treating them as reinforced concrete slabs, the procedure being as follows:

(a) The position of the neutral axis is found by assuming that the reinforcement is concentrated at the centroid of the steel joist, using expression (viii), Chap. XIX.

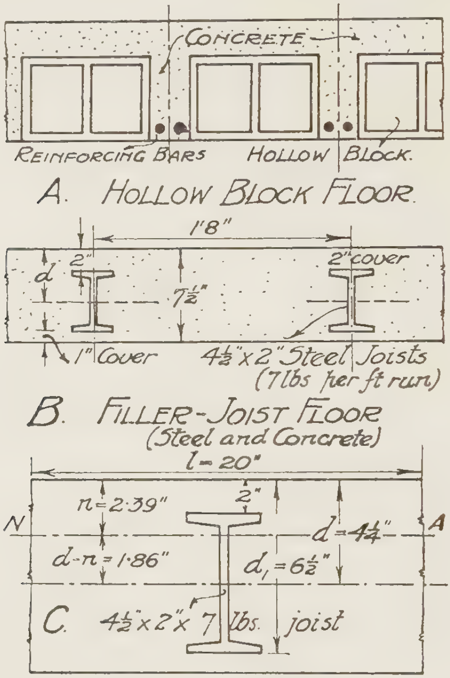


Fig. 235. Types of reinforced floor slabs.

(b) The equivalent moment of inertia (I_e) is then found by the method explained in para. 177 (A), adding, in this case, the moment of inertia of the steel joist about its own axis (I_x).

(c) The safe bending moment is found from expression (xi),

$$B = R = \frac{cI_e}{n}.$$

(d) The stress in the steel may then be checked by means of a modification of expression (xii),

$$R = \left(\frac{t}{m}\right)\frac{I_e}{d_1 - n},$$

where d_1 is the depth to the underside of the steel joist. The following Example will further explain the procedure.

Example. *Find the safe load per super ft. which can be put upon the "filler-joist" floor, detailed in Fig. 235 (B) and (C). The span is 10 ft.*

From Appendix I we have

$$A_t = \text{area of joist} = 2\cdot06 \text{ sq. ins.},$$
$$I_x = 6\cdot65 \text{ inch units}^4.$$

In this case d, the effective depth, is 4·25 ins.

If N.A. be the neutral axis, then we have in effect a T-beam and we can settle the width of the beam b on the usual basis. We will make b equal to the distance between the joists, that is 20 ins. Then

$$r = \text{ratio of reinforcement} = \frac{A_t}{bd} = \frac{2\cdot06}{20 \times 4\cdot25} = 0\cdot0242.$$

Using expression (viii), Chap, XIX, we have

$$\frac{n}{d} = \sqrt{2mr + m^2r^2} - mr$$
$$= \sqrt{30 \times 0\cdot0242 + 15^2\,(0\cdot0242)^2} - 15 \times 0\cdot0242$$
$$= 0\cdot562,$$

whence $\qquad n = 0\cdot562d = 0\cdot562 \times 4\cdot25 = 2\cdot39$ ins.

The equivalent moment of inertia (I_e), see para. 177 (A),

$$= \frac{bn^3}{3} + mI_x + mA_t\,(d - n)^2$$
$$= \frac{20 \times 2\cdot39^3}{3} + 15 \times 6\cdot65 + (15 \times 2\cdot06 \times 1\cdot86^2)$$
$$= 298 \text{ inch units}^4.$$

Then $\qquad R = \dfrac{cI_e}{n} = \dfrac{600 \times 298}{2\cdot39} = 75{,}000$ lb. ins.

To check stress in steel (t) we write (as explained above)

$$R = \left(\frac{t}{m}\right)\frac{I_e}{d_1 - n} = \frac{t \times 298}{15\,(6\cdot5 - 2\cdot39)} = \frac{298 \times t}{15 \times 4\cdot11} = 75{,}000 \text{ lbs.},$$

whence $\qquad t = \dfrac{75{,}000 \times 4\cdot11 \times 15}{298} = 15{,}500$ lbs. per sq. in.

Now, if W be the total distributed load carried by one joist, then

$$B = \frac{WL}{8}, \text{ or } 75{,}000 = \frac{W \times 120}{8},$$

whence $W = 5000$ lbs.

The area over one joist equals 10 ft. by 1·67 ft. Therefore

$$\text{Total load per foot super} = \frac{5000}{10 \times 1\cdot67}$$
$$= 300 \text{ lbs.}$$

The weight of a foot super of the floor slab, at 150 lbs. per cu. ft.,

$$= \frac{150 \times 7{\cdot}5}{12} = 94 \text{ lbs.}$$

Therefore the safe additional load which can be carried by the floor

$$= 300 - 94 = 206 \text{ lbs. per ft. super.}$$

Bending moment for continuous beams and slabs. In the case of beams and slabs which are continuous over more than two equal bays and are to be uniformly loaded, then, if more accurate information as to the exact bending moments to be resisted is not available the following approximation (based on the L.C.C. regulations) may be used:

Approximate bending moments on continuous beams and slabs with uniform loading over equal spans.

(*a*) Near the middle of the end spans take $B = \frac{wL}{10}$,

(*b*) At each intermediate support take $B = -\frac{wL}{10}$,

(*c*) At the middle of each interior span take $B = \frac{wL}{12}$,

where w is the load per foot of beam or slab, and L is the span over each bay.

179. Compression reinforcement. When a beam is continuous over a support we know that it will be subjected to a negative bending moment; see Chap. XVII. In order that a reinforced concrete beam may resist this "reverse" bending moment, some of the bottom bars may be bent up to the top of the beam, as in Fig. 240, and others added if necessary. In such a case there will be reinforcing bars at both the top and bottom of the beam, hence it is described as a **doubly reinforced beam.**

Again, if we examine Fig. 230, it will be seen that as r, the ratio of reinforcement, is increased beyond the economic ratio of 0·00675, the strength of the beam—which varies with k—does not increase at a corresponding rate; thus the increase of r to 0·02 only increases k from 95 to 131. As we have already seen this slow increase is due to the fact that at these ratios the steel is not being fully stressed. It is possible to increase the stress in the tensile reinforcement, and so make it more effective, by adding steel to the compression side, since this addition, by increasing the force C, adds to the total value of T. The complete treatment of such sections is difficult but, where a section is given, it is possible to ascertain its strength by the following approximate methods.

(A) **Doubly reinforced beams. Approximate treatment by the equivalent section method.** It ought not to require further explanation to show that the equivalent section in the case of a doubly reinforced beam will be as shown in Fig. 236, in which an area equal to $(m - 1) A_c$ is added to the compression area on the same level as the compression reinforcement, where A_c is the total area of the compression reinforcement. The area A_c is multiplied by $(m - 1)$, since this area, together with the space occupied by the bars, that is A_c, and now supposed to be occupied by concrete in the equivalent section, will give a total of

$$(m - 1) A_c + A_c,$$

that is mA_c. The equivalent tensile area will as before be mA_t.

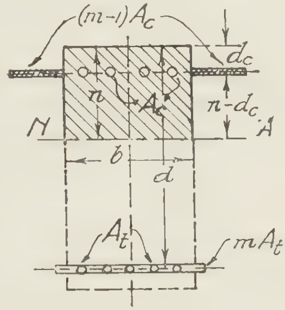


Fig. 236. Doubly reinforced beam—equivalent section.

By means of the methods explained in para. 177 (A) we first find the position of N.A. the neutral axis, and second the value of I_e, the moment of inertia of the equivalent section. Using this value for I_e we can, as explained in para. 177, either find R and c if t is given, or find c and t if R is given.

To find the stress in the compression reinforcement we need only remember that in this case it is limited to m times the stress in the surrounding concrete, so that if, by means of the expression, $R = fI/y$, we calculate the stress in the concrete at the level of the compression reinforcement, then the stress in the reinforcement will be m times this amount. As will be seen, from the value worked out in the next sub-paragraph, this stress is not high. Even if the reinforcement lay *in* the compression edge—which is of course impossible—it could not exceed $(m \times 600)$ or 9000 lbs. per sq. in.

(B) **Doubly reinforced beams. Approximate lever-arm method.** The following approximate method, which is somewhat similar to the second method already described for dealing with T-beams, see para. 177 (B), is somewhat easier than that given above and is equally suitable for checking or design; see Fig. 237.

(i) The work is divided into two main sections. In the first the beam is considered as a singly reinforced beam with r having the value 0·00675, from which the area of the tensile reinforcement is calculated.

(ii) The lever arm in a beam having r equal to 0·00675 has the value $(d - n/3)$, where n has the value $0{\cdot}36d$, which gives the **lever arm** $= a = 0{\cdot}88d$. *This value is used in the calculations, and both the "centre of compression" and the compression reinforcement*

*is assumed to be at this distance from the tensile reinforcement.**

(iii) The total tensile reinforcement A_t can now be found from the expression $R = aT = a(A_t . t)$: (or R can be found if A_t is given).

(iv) The *excess* of the tensile reinforcement (A_t) over $r = 0{\cdot}00675$, see (i), is next balanced against the compression reinforcement so that, (excess tensile reinforcement) × 16,000 = (area of compression reinforcement) × (stress in compression reinforcement).

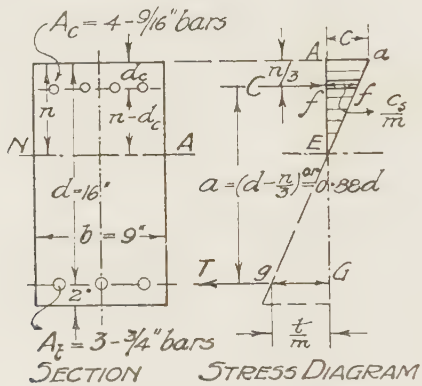


Fig. 237. Doubly reinforced beam.

(v) The stress in the compression reinforcement will be m times that in the concrete at the same level. From (i) the stress in the concrete reaches the full value of 600 lbs. per sq. in. at the compression edge. Since the compression reinforcement is placed at a distance of $\frac{2}{3}n$ from the neutral axis, see Fig. 237, the stress in it will be $(600 \times \frac{2}{3})\, m = 6000$ lbs. per sq. in.

As the following worked example will show, this method is not so difficult as might appear from the explanation; it is, however, essential that the reader should understand the simple principles upon which the method is based.

Example. *A beam, see Fig.* 237, *is to be* 9 *ins. wide and* 16 *ins. effective depth and is to carry a bending moment of* 300,000 *lb. ins. Find the dimensions of the necessary tensile and compressive reinforcement.*

In this case d = 16 ins. Hence

$$\text{approx. lever arm} = 0{\cdot}88 \times 16$$
$$= 14 \text{ ins. very nearly.}$$

* In practice the compression reinforcement is frequently placed at a distance of 2 ins. from the compression edge. This simplifies the calculations still further, if method (B) is used, since in this case $a = (d - 2)$ ins.

Hence $\quad T$ = total tensile force $= \dfrac{R}{a} = \dfrac{300{,}000}{14} = 21{,}400$ lbs.

$$= 16{,}000 \times A_t.$$

Whence $\quad A_t$ = area of tensile reinforcement $= \dfrac{21{,}400}{16{,}000}$

$= 1{\cdot}34$ sq. ins., say three $\frac{3}{4}$ in. round bars,

which gives $(3 \times 0{\cdot}442)$, or $1{\cdot}326$ sq. ins.

Now for $r = 0{\cdot}00675$,

$$A_t = 9 \times 16 \times 0{\cdot}00675 = 0{\cdot}97 \text{ sq. in.}$$

Therefore excess tensile reinforcement over that required for $r = 0{\cdot}00675$ is $(1{\cdot}33 - 0{\cdot}97)$, or $0{\cdot}36$ sq. in. The total stress on this area in tension will be $(0{\cdot}36 \times 16{,}000)$, or 5760 lbs. To provide this force in compression, at a stress of 6000 lbs. per sq. in., we will require compression reinforcement having an area

$$= \frac{5760}{6000} = 0{\cdot}96 \text{ sq. in.},$$

say four $\frac{9}{16}$ in. round bars, which gives $(4 \times 0{\cdot}249)$, or $0{\cdot}996$ sq. in. These may be placed 2 ins. below the compression edge; see Fig. 237.

The "tying-in" of compression reinforcement. In order to prevent compression bars buckling, and so causing the beam to fail in this way, adequate "cover" must be given to these bars. They ought, in addition, to be "tied-in" by means of "hoops" or "stirrups"; the stirrups used to resist the shear stresses are usually carried over the compression bars for this purpose and others may be added.

180. Adhesion between concrete and steel. As we have seen, the effectiveness of a flitch beam depends upon the way in which the two materials are strained together. In the case of reinforced concrete this is ensured by the "grip" which the concrete exerts upon the steel, and which is probably due in the main to the slight contraction of the concrete on setting.

The magnitude of this grip, or "adhesion" as it is often called, is not easily obtained experimentally, but it is usual to allow a *safe adhesive or grip stress of* 100 *lbs. per sq. in. of the surface of the bars*, provided that the ends are hooked to a standard form (see below).

Grip length. It will be obvious that the full strength of the bars can only be exercised when they are firmly held in the concrete; the length which a bar must be embedded before this strength is reached is known as the "grip length". We may easily find this length for the given values of 100 lbs. per sq. in. grip stress and 16,000 lbs. per sq. in. tensile stress in the steel.

Total pull on bar = total adhesion over the grip length (l),

or $$16{,}000 \times \pi \times d^2/4 = 100 \times \pi \times d \times l,$$

from which we have

l = grip length = $40d$ = 40 times the diameter of the bar.

In addition to this grip length the ends of bars are split, turned-up or "hooked" to assist them in carrying their full load as already mentioned; see Fig. 240.

181. Adhesion in a beam. If we take two sections in a reinforced concrete beam which are a short distance x apart, see Fig. 238, the bending moment at the first section AB being B_1 and that at the second section CD being B_2, then the tension in the reinforcement at AB will be $T_1 = B_1/a$ and that at CD will be $T_2 = B_2/a$, where a is the lever arm.

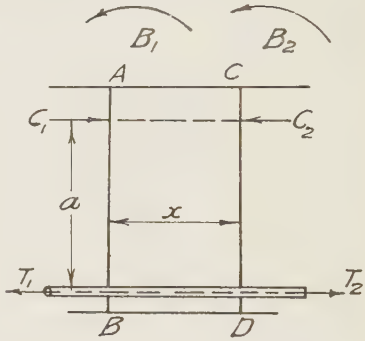


Fig. 238. Adhesion in a beam.

Now the difference in pull between these two tensions, which is given by $T_1 - T_2$, that is by

$$(B_1 - B_2)/a,$$

is the force which is communicated from the concrete to the steel by the grip stress over the length x. If therefore this stress be f' and there are n bars of a diameter d, the total area over which this stress acts will be $(xn\pi d)$. Hence

$$f'\,(xn\pi d) = \frac{B_1 - B_2}{a},$$

or

$$f' = \frac{B_1 - B_2}{x} \times \frac{1}{n\,(\pi d)\,a}.$$

But $(B_1 - B_2)/x$ is the rate of change of the bending moment, which equals the shear force (S) over this section, see para. 92. From this we have

$$f' = \frac{S}{n\,(\pi d)\,a}. \qquad \text{......(xiii)}$$

In any particular case this should give a grip stress below the safe stress (100 lbs. per sq. in.).

Example. *Find the maximum grip stress in the beam described in the last Example and shown in Fig. 237, if the beam spans an opening of 15 ft., the load being uniformly distributed.*

$$B = 300{,}000 \text{ lb. ins.} = \frac{WL}{8} = \frac{W \times 15 \times 12}{8}.$$

From which we have

$$W = \frac{300{,}000 \times 8}{15 \times 12} = 13{,}300 \text{ lbs.}$$

Hence the maximum shear which occurs at each end is

$$\frac{13{,}300}{2}, \text{ or } 6650 \text{ lbs.}$$

There are three $\frac{3}{4}$ in. bars and $a = 14$ ins., so that, inserting these values in the expression derived above, we have

$$\text{Adhesive stress} = f' = \frac{S}{n\,(\pi d)\,a} = \frac{6650}{3 \times \pi \times 0{\cdot}75 \times 14}$$

$$= 67 \text{ lbs. per sq. in.,}$$

which is well within the allowable stress.

182. Shear stress in reinforced concrete beams. The subject of shear stress in these beams is a difficult problem to investigate and in practice certain assumptions are made which only approximate to the true state of affairs. It is therefore reasonable that in this volume we should only attempt to give a straightforward treatment of the methods usually adopted.

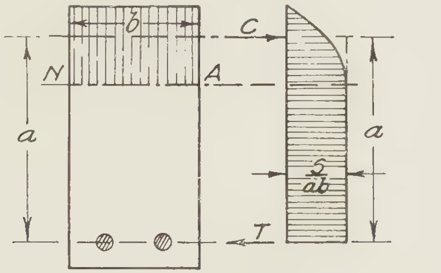


Fig. 239. Distribution of shear stress.

Apart from the action of the reinforcement it is clear that the distribution of shear stress for a plain rectangular beam would be parabolic; see para. 137. The effect of the adhesive stresses is, however, to develop the full shear strength of the concrete from the level of the tensile reinforcement up to the neutral axis, so that the shear stress, which is assumed to reach an intensity equal to S/ab, where a is the lever arm and b the thickness of the beam, is taken to be uniform up to the neutral axis, from which it dies away to zero at the upper edge; see Fig. 239.

This shear stress (s) *is generally limited to a safe working value of* 60 *lbs. per sq. in.*, so that *the safe shear force which the concrete will carry is* (60ab) *lbs.* (In the case of T-beams b_r should replace b.)

If this stress is exceeded, shear reinforcement must be provided. This can either be calculated to take (i) the margin of the total shear force, that is $(S - 60ab)$, or (ii) the whole of the shear force. The latter is the more logical and simpler plan and is generally adopted in British practice.

183. Inclined shear reinforcement. In freely supported reinforced concrete beams, since the magnitude of the bending moment falls off towards the supports, some of the tension bars may become unnecessary and can be bent up as shown in Fig. 240. This, as we shall presently see, assists the beam to resist the shear forces, which of course reach their maximum values near the points of support.

Very briefly we may say that such an arrangement may be regarded as forming a "trussed beam", in which the inclined bar is in tension, while the concrete, between the upper bend and the lower "hook", is in compression; see Fig. 240.

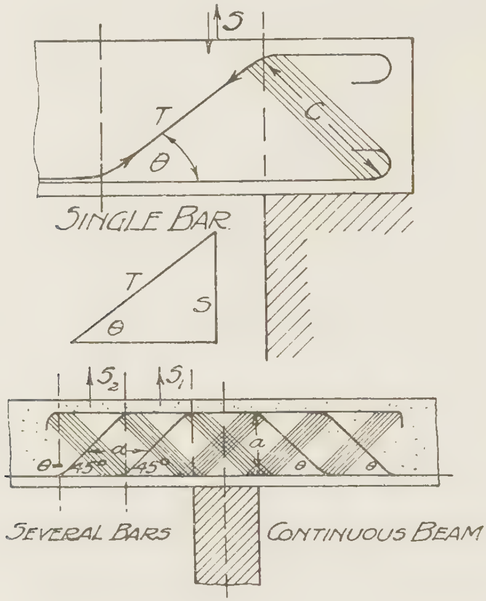


Fig. 240. Shear reinforcement—inclined bars.

If the inclined bar is to resist the whole of the shear force (S) acting over this section—($S - 60ab$) may be taken if desired—it is clear that the tensile force acting in it will be equal to the inclined component of S (see the small force diagram in Fig. 240, and refer also to Chap. XI, and para. 97), so that

$$\text{Force in bar} = T = \frac{S}{\sin\theta},$$

where θ is the inclination of the bar to the horizontal.

If A_s is the total area of this inclined reinforcement and t the stress therein, then

$$T = tA_s = \frac{S}{\sin\theta},$$

or

$$S = tA_s \sin\theta. \qquad \text{......(xiv)}$$

If more than one bar is turned up, see second sketch in Fig. 240, the same procedure is followed, the proper value for S over each "bay" being used in each case.

The usual value for θ is 45°. Where several bars are turned up, the spacing should not exceed a, the lever arm, otherwise the horizontal component of the compressive force in the concrete, which has to be resisted by the adhesive forces round the bars, might become excessive.

184. Vertical shear reinforcement. If there are no inclined bars, or if those provided are insufficient to resist the shear force acting on the beam, then vertical shear reinforcement may be provided in the form of "stirrups"; see Fig. 241.

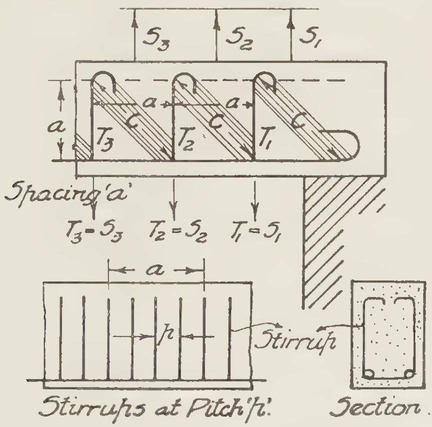


Fig. 241. Vertical shear reinforcement.

In this case again the spacing should not exceed a, the lever arm. Once more we may base our design upon the conception of a "truss" formation, so that, in this case, we have a truss with vertical tension members —the stirrups —and inclined compression members, in which the compressive forces are taken by the concrete. As explained in Chap. XI the forces acting in the vertical members will equal the vertical components of the forces acting in the inclined members, and these, in turn, will be equal to the shear forces acting over these "bays"; see Fig. 241. Hence, very simply,

if T *be the force in any vertical and* S *be the corresponding shear force, we have* $S = T = tA_s$, *where* A_s *is the sectional area of a single stirrup.* ......(xv)

It is frequently convenient to use stirrups of small section spaced at frequent intervals, the spacing of which may vary roughly with the intensity of the shear force along the length of the beam.

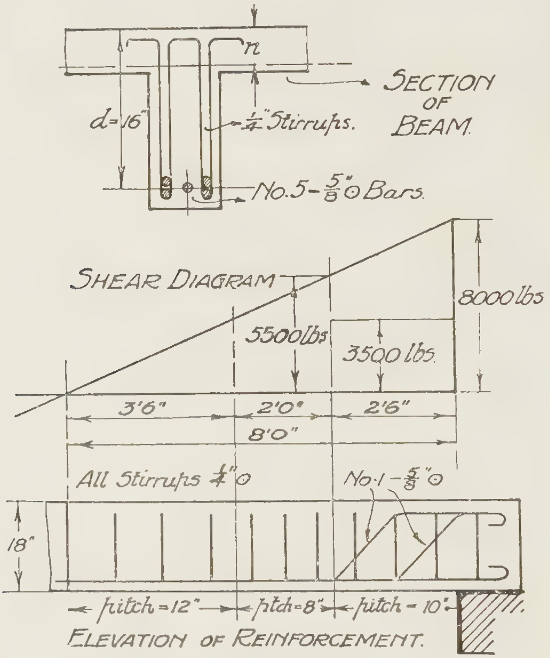


Fig. 242. Shear reinforcement.

Let p be the "pitch" or distance between two stirrups, see Fig. 241, so that, in the distance a, if there are n stirrups required, then $n = a/p$. Also let R_s be the resistance of one stirrup, that is (stress × total sectional area of stirrup). If we assume that over any length a the shear force is uniform and equal to S, then we have

$$S = nR_s = \frac{aR_s}{p},$$

from which we have

$$p = \text{pitch of stirrups} = \frac{aR_s}{S}. \qquad \text{......(xvi)}$$

(The similarity between this expression and that given in para. 164, to find the pitch of rivets in a compound girder, should be noted.)

The following example will suggest the procedure which may be followed in designing shear reinforcement.

Example. *Taking the* T*-beam section shown in Fig.* 242, *of a freely supported beam carrying a uniformly distributed load of* 16,000 *lbs. over a span of* 16 *ft., find the dimensions of suitable shear reinforcement,* (A) *if* (b_r), *the thickness of the rib, is* 8 *ins., and* (B) *if the rib is* 6 *ins. thick.*

(A) The shear stress near the supports will be $s = S/ab_r$, see para. 182, where S is 8000 lbs. By calculation n is 3·43 ins., so that a, which is equal to $(d - n/3)$, is $(16 - 3{\cdot}43/3)$, or 14·86 ins., and b_r is 8 ins. Therefore

$$s = \frac{8000}{14{\cdot}86 \times 8} = 67 \text{ lbs. per sq. in.}$$

This is very near the safe stress but may be allowed if we turn up one of the $\frac{5}{8}$ in. bars near the support.

The shear strength of one $\frac{5}{8}$ in. round bar turned up at an angle of 45°, see Fig. 242, is given by:

$$\begin{aligned} S &= tA_s \sin\theta \\ &= 16{,}000 \times 0{\cdot}307 \times 1/\sqrt{2} \\ &= 3500 \text{ lbs. nearly.} \end{aligned}$$

Hence this arrangement would make the beam safe.

(B) If the rib of the beam is only to be 6 ins. thick it will be advisable to ignore the shear strength of the concrete and provide shear reinforcement to take all the shear.

We may first turn up two of the bars singly at intervals equal to a, or say 15 ins. for convenience. This will account for 3500 lbs. of the shear over the first 30 ins. of the beam—see the shear force diagram in Fig. 242—leaving a maximum value of (8000 – 3500) or 4500 lbs. to be provided for by vertical stirrups.

If we use two U-shaped stirrups in combination, see section in Fig. 242, the stirrups being made from $\frac{1}{4}$ in. round bar, the strength of one such combination will be

$$R_s = 4 \times 16{,}000 \times 0{\cdot}049 = 3140 \text{ lbs.}$$

Hence $$p = \text{pitch} = \frac{aR_s}{S} = \frac{14{\cdot}86 \times 3140}{4500} = 10{\cdot}4 \text{ ins.,}$$

or say the pitch over the first 30 ins. is 10 ins.; see Fig. 242.

At 30 ins. from the edge of the support the shear force from the diagram is 5500 lbs. Over the next 2 ft. we may therefore use a closer pitch,

$$p = \frac{14{\cdot}86 \times 3140}{5500} = 8\tfrac{1}{2} \text{ ins., or say 8 ins.}$$

Beyond this section the pitch may be opened out to say 12 ins.; the calculations are left as an exercise for the reader.

185. Experiment—Tests on reinforced concrete beams.

Introductory tests. As an experimental introduction to the purpose and effect of reinforcement in a concrete beam, a test may be carried out upon two (or more) beams made up from 1 : 2 : 4 concrete and tested

at the end of 28 days. The first beam (*a*) should be made up from plain concrete, while the second (*b*) should be reinforced to resist tension only, by bars placed near the bottom edge.

Test beams. While the beams should be as large as can be conveniently made up and tested, quite good results may be obtained with relatively small beams. A section 5 ins. by 3 ins. is suitable, with two $\frac{1}{4}$ in. round bars placed 1 in. from the lower surface. Such a beam would require a central load of from 800 to 1200 lbs. to fracture it over a span of 5 ft., so that the test does not necessitate the use of a powerful testing machine.

Beam (*a*) will of course fail by tension on the under surface, while beam (*b*) will probably fail, at a much higher load, by the shearing of the concrete near the ends (shown by an inclined fracture).

Shear reinforcement. Other beams should then be designed with suitable reinforcement to resist both tension and shear, to be tested preferably at the age of 4 months. *The period can be shortened by the use of a rapid-hardening cement, in which case, however, other working factors may have to be used in the calculations.*

It will be an advantage to design these beams for two-point loading, see Fig. 207, as in this way the maximum shear stresses will be restricted to the end portions of the beam.

Deflection tests. Before these latter beams are tested to destruction, some deflection tests may be carried out in the manner described in Experiment, para. 149. The values so obtained may be compared with calculated values (also found as explained in para. 149). The value of I_e, the equivalent moment of inertia of the section, may be used in making these calculations.

The experimental and calculated values for deflection should approximate to each other, at loads which produce stresses in the beams corresponding to those used in their design.

Test to destruction—Factor of safety. Each beam may finally be tested until it fails, the maximum load and manner of failure being carefully noted. The factor of safety may be found from

$$\text{Factor of safety} = \frac{\text{breaking load}}{\text{calculated load}},$$

and will usually give values of from 3 to 3·5.

(**Note.** To carry out this latter test properly, experiments to obtain the ultimate strength of the concrete, see para. 206, and also to find E for the concrete and the steel, should be conducted. Suitable adjustments must then be made in the values used for calculating the strength of the beam. Such experimental refinements are, however, only possible where adequate testing facilities are available.)

Problems XX

1. A reinforced concrete T-beam has a total depth of 20 ins., while the breadth of the slab may be taken to be 36 ins., and the thickness 6 ins. The thickness of the rib is 8 ins. Find the value of R and the maximum stresses in the steel and the concrete, if the beam is reinforced with four 1 in. round rods.

2. If the slab in Prob. 1 had only been 4 ins. thick, what would have been the value of the equivalent moment of inertia and of the moment of resistance (R)? (Use the method of para. 177 (A).)

3. If in the beam described in Prob. 1 the total depth had been increased to 30 ins., what reinforcement would have been necessary to raise the value of R to 1,750,000 lb. ins.? (Use the method described in para. 177 (B).)

4. Calculate the equivalent moment of inertia of the doubly reinforced beam shown in Fig. 237. Find the value of R, the stress in the concrete and in the compression reinforcement, when the tensile reinforcement is stressed at 16,000 lbs. per sq. in.

5. Calculate suitable tensile and compression reinforcement for a beam 15 ins. wide and 25 ins. total depth, to give a moment of resistance (R) equal to 1,000,000 lb. ins.

6. Find the adhesion or grip stress on the bars near the ends of the beam shown in Fig. 234 (R = 921,600 lb. ins.), if the load is uniformly distributed over a span of 15 ft.

7. If the rib of the beam shown in Fig. 234 is 8 ins. thick, what would be the value of the shear stress in the concrete near the ends of the beam (when loaded as described in Prob. 6) if no shear reinforcement were to be used?

If one of the $1\frac{1}{4}$ in. reinforcing rods is turned up at an angle of 45° near the ends of this beam, what total shear force will it be capable of resisting?

8. If in the beam shown in Fig. 242 no inclined shear reinforcement is used, calculate the pitch of the $\frac{1}{4}$ in. vertical stirrups near the ends of the beam.

CHAPTER XXI

MASONRY OR BLOCK CONSTRUCTION—WALLS

186. Masonry. The word "masonry" will be used in this and the succeeding chapters in its more general sense, as applied to all forms of construction in which **blocks of material**—stone, brick, concrete, etc.—are used to form a structure.

Masonry joints. The common practice in forming joints between these blocks is to lay a bed of lime or cement mortar in the joint. So far as we are here concerned with the question of the nature of the joints, we may regard this method merely as a ready means of ensuring that each block beds solidly upon the next below it, so as to enable it to transmit effectively any load which may come upon the upper block.

So far as the *compressive strength* of a wall or pier is concerned this will usually be limited by the strength of the mortar, since the latter will in most cases be less than that of individual blocks (see working stresses given below in Table X).

Table X

Working stresses on Masonry and Foundations (see para. 188)

Material	Stress: tons per sq. ft.	Stress: lbs. per sq. in.	Weight: lbs. per cu. ft.
Masonry			
Ashlar, in cement mortar	12	180	140–160
Blue brick, in cement mortar	12	180	140
Pressed red brick (engineering brick), in cement mortar	8	120	130
Stock brick, in lime mortar	5	80	120
Concrete, in walls (1 : 2 : 4)	38	600	140
Concrete, in foundations (1 : $2\frac{1}{2}$: 5)	12	180	135
Foundations			
Rock	8–12	120–180	—
Firm gravel and hard blue clay	4	60	100–120
Clay and firm earth	2	30	120–135
Unconfined sand and loose earth	1	15	120

The *adhesive* or *tensile strength* of such joints as are here considered is so small and so uncertain that we shall as a rule disregard its effect altogether, so that *we may look upon the joints in masonry structures as being incapable of transmitting tensile forces.*

We thus arrive at a conception of a **masonry structure** as being *one in which blocks merely rest one upon the other*—as in a wall—or in which one block is pressed strongly against another—as in an arch. It is upon this conception as a basis that we shall build up our theory of masonry construction, our chief concern being, as we shall see, with the action taking place at the joints.

187. Loads on masonry joints. All the loads with which we shall have to deal may be classified in the following manner:

I. **Vertical or normal loads.** All these act at right angles to the surface of the joint; they may be subdivided into:

(a) **Central or axial loads.** See Fig. 243 (i); these act at the centre of the joint, or more strictly at G the centroid of the area of the joint; see Fig. 243 (v).

(b) **Eccentric loads.** See Fig. 243 (ii); these act at a distance, say e, from the centre or centroid of the joint. The distance e may be termed the "eccentricity" of the load P.

II. **Horizontal loads or shear forces.** See Fig. 243 (iii); these act horizontally or are tangential to the surface of the joint.

It will be observed from Fig. 243 (iv) that **inclined loads** may be treated as a combination of vertical and horizontal loads, which may or may not be eccentric. We are thus able to divide our treatment into two main portions, one dealing with vertical loads, and the other with horizontal and tangential loads.

Since it is usual to indicate the forces acting upon a wall by means of a sectional drawing of the wall, it will be convenient to call the dimension parallel to the paper the "depth" (d) of the joint, and the dimension at right angles to the plane of the paper—measured along the length of the wall—the breadth (b) of the joint, see Fig. 243 (v) and compare with the lettering of beam sections in Chap. XIII.

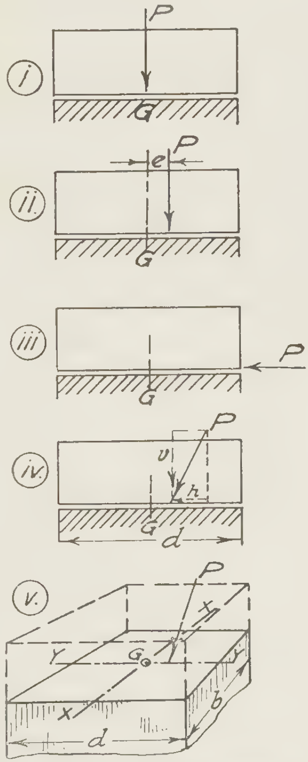


Fig. 243. Loads on masonry joints.

188. Central or axial loads. In the case of a load which acts at the centre of a rectangular section (or at the centroid in the case of differently shaped sections), the distribution of pressure is taken to be uniform over the surface of the joint. The intensity of pressure, or of compression (c), will therefore be given by

$$\text{Intensity of pressure} = c = \frac{\text{load}}{\text{area of joint}} = \frac{P}{bd} \text{ (for a rectangle).} \qquad \text{......(i)}$$

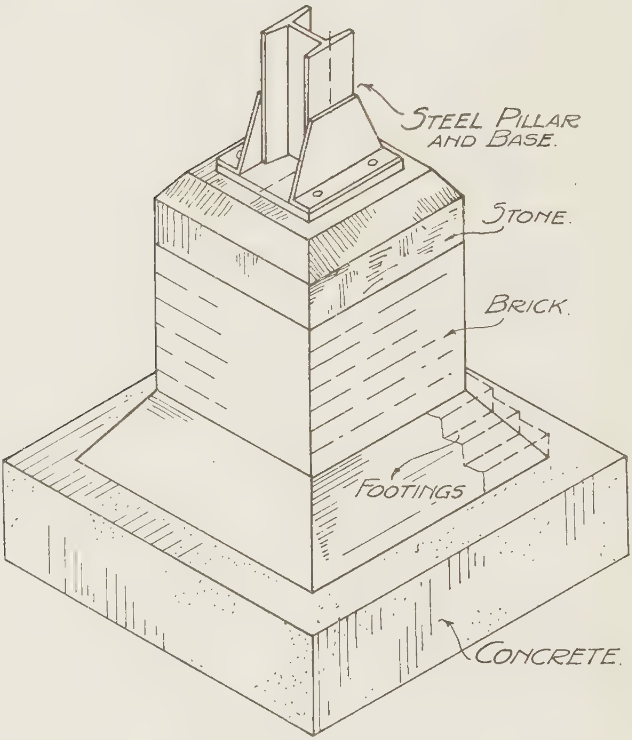


Fig. 244. Pier with footings, etc.

In such cases the only condition necessary for safety is that c should not exceed the safe working stresses which can be allowed. The values given above in Table X are those commonly used in practice. For ease of comparison the loads have been expressed (in round figures) both as "tons per sq. ft." and as "lbs. per sq. in." Either value may be used as found convenient, the former being most common in practice.

In an example such as that shown in Fig. 244—where a steel stanchion rests on a stone template, which is in turn supported by a brick pier, footings, a concrete bed and, finally, the earth foundation on which the whole structure is built—the problem resolves itself into ensuring that, at each joint, the area is such that the stress is brought within the safe stress of the *weaker material.* See Problems XXI, 1.

189. Eccentric loads. Let us now consider the effects produced when a vertical load P does not act at the centre of the joint.

It may be stated here that the following theory, which, in a somewhat less simple form, is that almost universally adopted in the design of masonry structures, is based on the following **Assumption: that the material of the blocks acts as an elastic material, strain and stress being proportional to each other within the working stress.**

The stress-strain curve for stone in compression is similar to that shown for concrete in Fig. 225, so that this assumption is only approximately true; it errs, however, on the safe side and gives satisfactory results in practice.

In the case of an axial load we have seen that the stress is given by $c = P/bd$, and is uniform over the area of the joint. We may therefore represent it by means of a diagram of stress; see Fig. 245 (i). Obviously the strain in this case will also be uniform and is represented by the dotted line parallel to fg, the line of the joint.

It should be noted that, if P' be the resultant of all the forces acting in the lower block to support the load P, then, for equilibrium, P and P' must be equal in magnitude and act in the same straight line. This is obviously so in this case.

If now the force P acts at any point which is not the centroid, as in Fig. 245 (ii), then neither the strain nor the stress will be uniform. It follows, however, from our Assumption that the strain, and therefore the stress, must *vary uniformly* from one point to another in the joint. Hence, if, in the diagram of stress shown in Fig. 245 (ii), the stress at f be c_1, while the stress at g is c_2, then the diagram of stress will be bounded by a straight line between these two values. The diagram of stress may thus be either trapezoidal, as in Fig. 245 (ii), or triangular, as in Fig. 245 (iii) or (iv). (The diagrams of strain will be similar in form to those for stress, see Fig. 245.)

Now in each of these cases, for equilibrium, P and P' must act in the same straight line. But we know that P' will act through the centroid G of the stress diagram. *Hence the centroid of the stress diagram lies in the line of action of* P *if* P *is vertical.*

Thus in Fig. 245 (iii), where the diagram is triangular, the stress at g being zero, the centroid G will be at a distance $d/3$ from f,

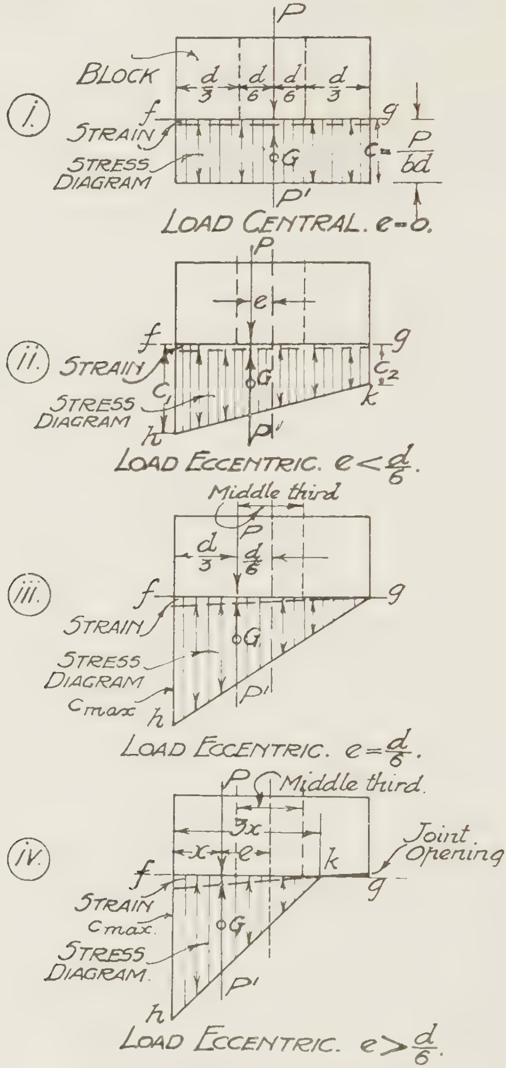


Fig. 245. Eccentric loads on masonry.

and therefore P' (and also P) will act at this distance from f. To express this in terms of the eccentricity (e) of the force, the eccentricity of the force P is $d/6$.

If the width d of a joint be divided into three equal parts, then the middle one is known as the **Middle Third**. The force P in the case just mentioned acts at the edge of the middle third.

If, as in Fig. 245 (iv), the force P acts outside the "middle third", so that the eccentricity is more than $d/6$, then, while we will still get stress varying from zero at some point k to a maximum at the edge f, the diagram of stress being triangular, the base of this triangle will be less than the width of the joint, being equal to $3x$, where x is the distance from the edge f to the force P. Over the remaining part of the joint, from k to g, there would be a tendency to set up tension in the mortar, but if, as we have assumed, the mortar is unable to resist tension, then the joint will tend to open as shown in Fig. 245 (iv).

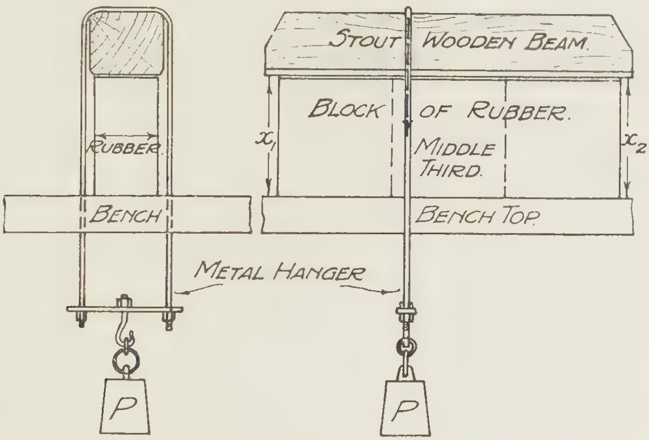


Fig. 246. Effect of an eccentric load on a block of rubber.

Experiment. The truth of the preceding statement can be readily demonstrated by means of the apparatus shown in Fig. 246. The large piece of rubber is to be taken as representing a block of masonry, to which is transmitted, through the wooden "block" above, a force P. The position of the force P can be varied as desired. Practically the whole of the strain takes place in the rubber block and may be roughly measured by taking the distances between the wooden beam and the bench at each end. Each of the cases mentioned above should be taken. If P is of considerable dimensions the opening of the joint, when P lies outside the middle third, will be easily seen.

Similar results may be obtained—particularly useful when dealing with the question of foundations—by placing a board on a carefully "struck" surface of sand, and then noting the impressions made in the sand when a heavy weight (P) is placed at different positions on the board.

The above is a very simple explanation of the principles under-

lying what is known as the **Middle Third Rule** in masonry, and which states:

Masonry Rule I. To avoid tensile stress across a rectangular joint in masonry the load must be applied within the middle third.

(For sections other than rectangular see note in connection with expressions (iv) and (v) in paragraph 190.)

190. Safe compressive stress. In the diagrams given in Fig. 245 it will be noted that, if P had the same value in each case, then there would be a steady increase in the value of the maximum compression (c), from case (i) to case (iv). If in any of these cases the maximum compressive stress (c) exceeded the safe working stress by so great an amount as to approach the ultimate strength of the stone or mortar in compression, then the structure would in all probability fail by crushing at the edge. We thus arrive at our second rule:

Masonry Rule II. The value of c, the maximum compressive stress, must not exceed the safe working stress of the material.

It will be convenient to deduce expressions which will enable us to calculate these maximum stresses.

(a) **To find the maximum compressive stress (c) when P lies outside the middle third.** See Fig. 245 (iv). As before, let x be the distance from P to the nearer edge f of the joint, that is the point of greatest compression. Then by the reasoning already given the base fk of the pressure triangle will be $3x$. If P be the load coming on a "length" of the joint equal to b, then the area of the joint under stress will be $(3x.b)$. Again if c be the maximum compressive stress, then the average stress will be $c/2$, and the total force on the area will be $(3x.b)\,c/2$: this gives us the value of P', which is equal to P, hence

$$P = \frac{3x.b.c}{2},$$

from which we have

$$c_{max.} = \text{max. compression } (P \text{ outside middle third}) = \frac{2P}{3x.b}. \qquad \text{......(ii)}$$

When P lies at the edge of the middle third, then

$$x = \frac{d}{3} \text{ and } c_{max.} = \frac{2P}{3b} \times \frac{3}{d} = \frac{2P}{bd}, \qquad \text{......(iii)}$$

or twice the mean intensity of stress (P/bd) *when* P *is central.*

(b) **To find the maximum compressive stress (c) when P lies within the middle third.** See Fig. 247. An expression could be obtained in this case by reasoning along the lines adopted above. It will probably be more instructive, however, to adopt a more general treatment.

As indicated in para. 72, a single force may be balanced by a couple plus a single force. In the case illustrated in Fig. 247 therefore, P may be looked upon as being balanced by a single

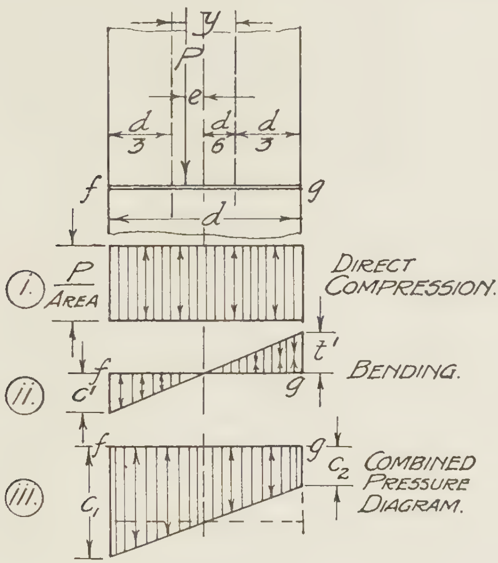


Fig. 247. Eccentric loads on masonry—(e) less than $d/6$.

force P', which acts at the centre (or centroid) of the joint, and is the result of a uniform compressive stress (P/area) over the surface of the joint, see Fig. 247 (i); and a clockwise couple (or bending moment) equal to the force P multiplied by the eccentricity e.

The bending moment Pe will produce a tensile stress (t') at one edge and a compressive stress (c') at the other; see Fig. 247 (ii). Using the expression $f = B/Z$, see Chap. XIII, we have that $c' = t' = Pe/Z$, where Z is the modulus of the area of the joint and the joint is symmetrical about an axis through the centroid.

Adding the diagrams (i) and (ii) algebraically we get the combined diagram (iii), from which we see that at f, the edge of maximum compression, the quantity (P/area) is increased by the

amount c', while at the opposite edge g it is decreased by an equal amount t'; see Fig. 247 (iii). We thus get the following general expressions:

$$\text{Maximum compression} = \frac{P}{\text{area}} + \frac{Pe}{Z}, \qquad \ldots\ldots\text{(iv)}$$

$$\text{Minimum compression} = \frac{P}{\text{area}} - \frac{Pe}{Z}. \qquad \ldots\ldots\text{(v)}$$

The expressions (iv) and (v) are of general application: they may be applied to joints other than rectangular, such as hollow rectangles (as in chimneys), solid and hollow circles, etc., if the appropriate value for the "area" and for Z is used in each case.

Note. *If the joints cannot resist tensile stresses, then the use of the expressions* (iv) *and* (v) *must be limited to those cases where no tension is induced at the joint—that is* e *must not exceed the value which reduces expression* (v) *to zero—otherwise the maximum value of* c *will not be given correctly.*

For rectangular joints expressions (iv) and (v) may be put into a more convenient form.

Maximum pressure at a rectangular joint. In this case we have

$$\text{Area} = bd, \text{ and } Z = \frac{bd^2}{6},$$

hence from (iv) we have

$$\text{Maximum compression} = c_{\text{max.}} = \frac{P}{bd} + \frac{Pe}{\frac{bd^2}{6}}$$

$$= \frac{P}{bd} + \frac{6Pe}{bd^2} = \frac{Pd}{bd^2} + \frac{6Pe}{bd^2}$$

$$= \frac{P(d+6e)}{bd^2} = \frac{6P\left(\frac{d}{6} + e\right)}{bd^2}. \qquad \ldots\ldots\text{(vi)}$$

It should be noted that *the term* $(d/6 + e)$ is equal to the dimension which has been marked y in Fig. 247, and *is the distance from* P *to the farthest "middle third point".*

When P lies *on* the middle third point, then $(d/6 - e)$ becomes equal to $d/3$; so that the maximum compression will be

$$\frac{6P \times d/3}{bd^2},$$

which equals $2P/bd$ as in expression (iii).

The above rules are applicable when calculating foundation pressures on soils of uniform resisting power. Examples showing the use of these expressions are included in the next few paragraphs.

191. Horizontal Forces—Friction. A consideration of the effect of horizontal forces on masonry may best be introduced by an experimental investigation of the effects of friction.

Experiment. If a block of wood having a plane surface is laid upon a similar surface which is horizontal, say the top of a bench, see Fig. 248,

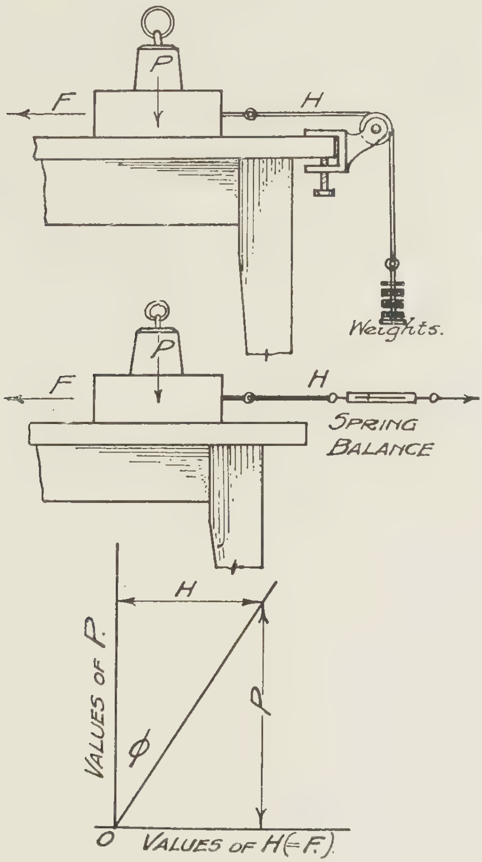


Fig. 248. Friction.

and a weight placed on the block, it will be found that the block will resist horizontal movement until the pulling force H reaches a certain value. The *force of friction*, which in this case resists motion, is therefore seen to act as though it were a force (F) pulling in the opposite direction to H, the applied force. This resisting force may evidently have a number of values, ranging from zero up to the value which it has when the block is just about to move under the action H. This latter value

is known as *the limiting value of* F, or merely the *limiting friction*, when F and H are just equal.

The magnitude of H, and hence of F, may be obtained experimentally by means of a spring balance, or by means of a cord passing over a pulley and carrying a series of weights; see Fig. 248.

A series of values of H for varying values of P should now be obtained and plotted in graph form as shown in Fig. 248. The values so obtained should lie approximately on a straight line, showing that in each case the value of H varies directly with that of P. We can thus say that

$$\boldsymbol{F} = \boldsymbol{H} = \text{limiting friction} = \text{constant} \times \text{weight} = \mu \boldsymbol{P}, \quad \ldots\text{(vii)}$$

where μ (*mu*) is known as the **Coefficient of Friction** for the two surfaces being considered.

The angle ϕ (*phi*), which is contained between the sloping line and the axis giving the values of P, is known as the **Angle of Friction.**

Experiment. The following statements should next be checked experimentally: (a) the value of F, the limiting friction, is not altered by an alteration of the area of the surfaces in contact; (b) the value of μ and of ϕ vary with different materials (forming the two surfaces) and different conditions of the surfaces (rough or smooth, etc.).

Values of μ and ϕ. The following values will be useful in applying these results to masonry and other structures.

Table XI

Coefficients and Angles of Friction

Surfaces in contact	Values of μ	Values of ϕ
Wood on wood	0·25 to 0·5	14° to 28°
Metal on metal	0·15 to 0·3	9° to 18°
Stone on stone	0·4 to 0·6	22° to 33°
Masonry on dry clay	0·5 (aver.)	28° (aver.)
Masonry on wet clay	0·3 (aver.)	16° (aver.)

192. Inclined forces on joints. If we now consider the effect of inclined forces at a masonry joint, see Fig. 249, it will be clear that, as already explained, the inclined force P may be divided into its vertical (V) and horizontal (H) components, and the effects of these may be considered separately. The component force H will tend to produce sliding, but it will be clear from our statement concerning the action of friction that no sliding will take place until H is equal to or greater than μV, or, to put it in another way, until the angle which P makes with a line at right angles to the joint is equal to or greater than the angle of friction (ϕ); see Fig. 249. We thus arrive at our third rule for masonry construction.

Masonry Rule III. To prevent sliding at the joints, the angle which the applied force P makes with a line normal to the joint must not exceed the angle of friction (ϕ) for the materials of the joint.

Having briefly set out the three rules which are to guide us in designing masonry which will be statically sound, we may now proceed to apply them in the solution of certain practical problems. As the reader will presently realise, apart even from certain complex questions involved in the design of very large masonry structures, such as arched bridges and dams to retain water, there are other difficulties arising out of any attempt to estimate accurately the forces which masonry structures have to resist, and the treatment which we shall be able to give to such questions here must therefore be looked upon merely as an introduction to these larger problems.

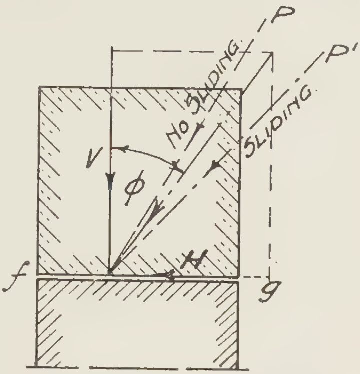


Fig. 249. Inclined forces on masonry.

193. Forces acting in a buttress. The procedure to be adopted in each case will probably be best explained by means of Examples.

Example. *The buttress shown in Fig.* 250 (*a buttress of these dimensions was dealt with in Prob.* 2, *Chap.* III) *is constructed of stone weighing* 140 *lbs. per cu. ft. and is* 3 *ft. thick. It is to be subjected to an inclined force* P *of* 7500 *lbs., acting at an angle of* 60° *to the horizontal, the line of action of which enters at a point* M *on the inner face of the buttress,* 1 *ft. above the first joint* A–A. *Find whether the resultant pressure, due to the force and the weight of each part of the buttress, falls within the middle third at each joint. In addition find the maximum compressive stress at the bottom joint* C–C.

The weight of each portion of the buttress is first calculated; this has been written alongside each portion; see Fig. 250.

The weight of each portion is taken to act at its centre of gravity as shown.

To find the resultant R' for forces P and W_1. See Fig. 250. The two forces P and W_1 intersect at the point d. The magnitude and direction of their resultant (R') can then be found by means of a parallelogram of forces. This gives the line do, which crosses the joint A–A just within the middle third.

To find the force R'' acting on the second joint B B. This force is clearly the resultant of the forces R' and W_2; it can be found by means of a parallelogram of forces. A method which is, however, preferable in such cases is to make use of a force diagram, shown in Fig. 250, in which the weight of each portion of the buttress is added successively to the original force P. The force R'' will act in a direction parallel to the

corresponding line in the force diagram and pass through the point e, where R' and W_2 intersect. The force R'' falls just within the middle third.

To find the force R''' acting at the joint C–C. Using the same procedure as before, the force R''' passes through the point f in which the forces R'' and W_3 intersect. This force again cuts across the joint at a point within the middle third.

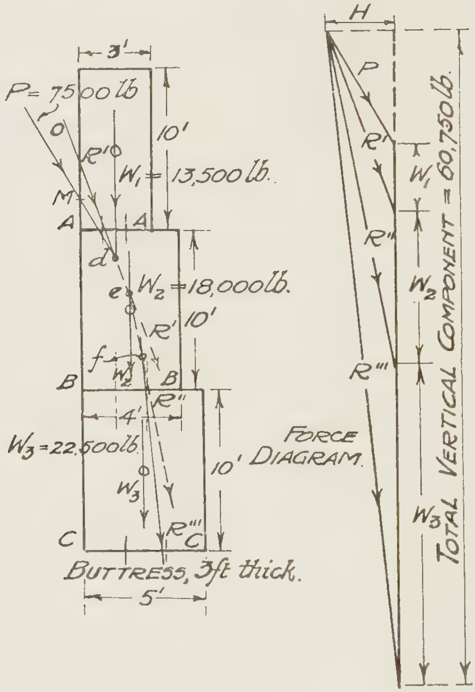


Fig. 250. Forces in a buttress.

To find the maximum pressure at joint C–C. R''' comes so near to the middle third point that we may assume that it acts at that point.

The vertical component of the total force R''' is, by measurement of the force diagram, 60,750 lbs. Then, from expression (iii), we have

$$c = \text{maximum compression} = \frac{2P}{bd} = \frac{2 \times 60{,}750}{3 \times 5} = 8100 \text{ lbs.},$$

or $3\frac{1}{2}$ tons (approx.) per sq. ft.

Sliding. At joint A–A, where the force acting on the joint has the *greatest inclination*, the angle to the vertical is only 20°; the buttress is therefore safe against sliding.

Line of Resultant Pressures or Line of Pressure. The line formed by the resultants R', R'' and R''' is sometimes called the **Line of Resultant Pressures** or simply the **Line of Pressure.** In this case it consists of several lengths of straight line. If, however, the effect of the weight of the buttress was added *at every point* in the height of the buttress, from the point M at which the force P entered, then the line of pressure would be a curved line. In the case of structures in which we only desire to know the effect of the loads at certain levels or at the joints, the approximate line of pressure, made up of straight lines and obtained as in the above example, will give accurate results.

194. Forces acting in a masonry dam. See Fig. 251. The work in this case may be done graphically, on the lines explained in the

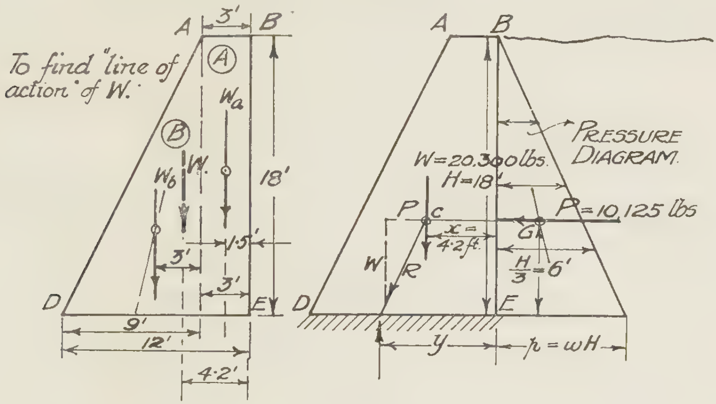


Fig. 251. Concrete dam—position of resultant forces.

preceding example, or partly graphically and partly by calculation as explained below. Since the stresses at only one joint are required the latter method is preferable. The centroid (or C.G.) of the dam is found by the methods explained in Chap. III.

Example. *Find the stresses at the line* DE *of the concrete dam shown in Fig.* 251, (a) *when the reservoir is empty, and* (b) *when the reservoir is full of water. The concrete may be assumed to weigh* 150 *lbs. per cu. ft.*

The total weight (W) of 1 foot lineal of the dam is

$$W = 18\left(\frac{12 + 3}{2}\right) 150 = 20{,}300 \text{ lbs. (approx.)}.$$

To find distance from E to line of action of W. If we divide the dam into portions (A) and (B), see left-hand figure, we may take moments about E, letting x be the distance to the line of action of W, the total weight. The weight of the portion A will be $3 \times 18 \times 150$, or 8100 lbs.

approx. Then the weight of portion B will be (20,300 − 8100), or 12,200 lbs. approx.

Taking moments about E we have

$$Wx = 20{,}300x = 8100 \times 1{\cdot}5 + 12{,}200 \times 6, \text{ see Fig. 251,}$$

whence $x = 4{\cdot}2$ ft.

Pressure on the face of the dam. Since the pressures on the face of the dam will vary with the depth of the water and be normal to the surface upon which they act (see Vol. I, or any book on elementary hydrostatics), the pressure diagram will be a triangle as shown in Fig. 251. The maximum pressure (p) at E is equal to wH lbs. per sq. ft., where H is the total depth and w is the weight of a cubic foot of water ($62\frac{1}{2}$ lbs.).

The total pressure (P) will act through the centroid G of the pressure diagram and, since this is a triangle, P will act at a height of $H/3$, or 6 ft. from the base of the wall.

The average pressure is $wH/2$, and this acts over an area of the dam of (18 × 1) sq. ft., since we are taking 1 ft. of the length of the dam. Hence

$$P = \text{total pressure} = \frac{1}{2}(wH)\,H = \frac{wH^2}{2} \qquad \text{......(viii)}$$

$$= \frac{62\frac{1}{2} \times 18^2}{2} = 10{,}125 \text{ lbs.}$$

The resultant (R) of the two forces P and W will act along some inclined line as indicated in Fig. 251, but in ascertaining the compressive stresses acting on the joint DE we need only consider the vertical component of R, and this will be equal to W, as shown in the figure. Hence the resultant of all the vertical forces *supporting* the dam must have a vertical component acting upwards and equal to W; let this force (say W_R) act at a distance y from E. Taking moments about E we have for equilibrium

$$W_R \times y = W \times 4{\cdot}2 + P \times 6,$$

or
$$y = \frac{20{,}300 \times 4{\cdot}2 + 10{,}125 \times 6}{20{,}300} = 7{\cdot}2 \text{ ft.,}$$

so that R comes within the middle third.

Pressures on the base. (a) **When reservoir is empty.** See Fig. 252. As we have seen, the line of action of W, the weight of 1 ft. of the dam, will cross the joint at 4·2 ft. from g. Hence using expression (iv) we have

$$\text{Maximum compression} = \frac{P}{\text{area}} + \frac{Pe}{Z}.$$

In this case

$$P = W = 20{,}300 \text{ lbs.,}$$

$$e = (6 - 4{\cdot}2) = 1{\cdot}8 \text{ ft.,}$$

$$Z = \frac{bd^2}{6} = \frac{1 \times 12^2}{6} = 24 \text{ ft. units}^3.$$

Whence

$$\text{Maximum pressure} = \frac{20{,}300}{12} + \frac{20{,}300 \times 1{\cdot}8}{24}$$

$$= 1690 + 1520 = 3210 \text{ lbs. per sq. ft. approx.}$$

Similarly from (v) we have

$$\text{Minimum pressure} = \frac{P}{\text{area}} - \frac{Pe}{Z} = 1690 - 1520$$

$$= 170 \text{ lbs. per sq. ft.}$$

These values are shown in the pressure diagram in Fig. 252.

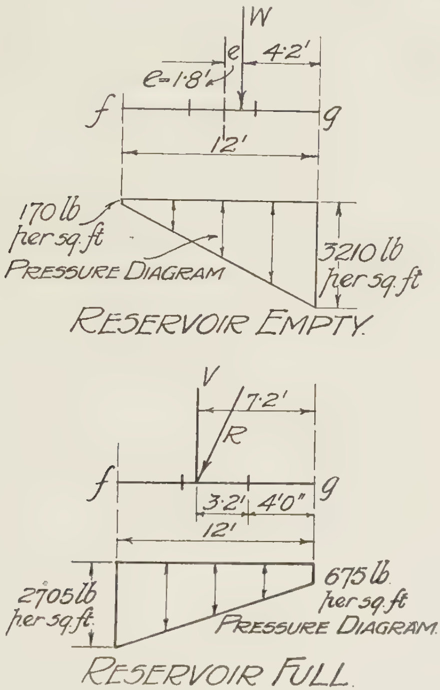


Fig. 252. Pressures on the base of a concrete dam.

(b) **When reservoir is full.** See Fig. 252. In this case the resultant force cuts the joint 7·2 ft. from E and its vertical component is 20,300 lbs. The value of e is (7·2 − 6), or 1·2 ft. Then, as before,

$$\text{Maximum compression} = \frac{20{,}300}{12} + \frac{20{,}300 \times 1{\cdot}2}{24} = 1690 + 1015$$

$$= 2705 \text{ lbs. per sq. ft.,}$$

$$\text{Minimum compression} = 1690 - 1015 = 675 \text{ lbs. per sq. ft.}$$

The horizontal force at the joint DE is of course 10,125 lbs., which is more than half W. Some method would therefore have to be adopted to prevent sliding. Usually such a wall would be continued to a greater depth than is shown here and bonded into a rock foundation (or other impervious stratum), which would provide resistance against sliding.

195. Forces acting in a retaining wall. The principal difficulty in the design of retaining walls is that involved in settling the nature of the pressures which the retained earth will exert upon the wall. Most of the theories which are in use agree in assuming that, as with water pressure, pressure from loose earth (*a*) varies with the depth, and (*b*) acts at right angles to the supporting face. Other factors which have to be considered are the weight (w) of a cubic foot of the earth which is to be retained. and the "angle of repose" for the same earth.

Angle of repose. If a steep bank of earth be exposed, it will gradually fall away to a slope AD, see Fig. 253, having a well-defined and fairly uniform inclination, which is known as the *angle of repose.* For most materials this angle is approximately equal to the angle of friction (ϕ). Common values of this angle, obtained by averaging results from experiment and by observation, are given in the following table.

Table XII

Weights of Soils and Angles of Repose

Material		Angle of repose (ϕ)	Weight per cu. ft.
Sand	Dry	30°–35°	90 lbs.
	Wet	26°–30°	120 lbs.
Earth	Dry	29°	90–95 lbs.
	Wet	17°	110–120 lbs.
Clay	Dry	29°	120 lbs.
	Wet	16°	135 lbs.
Gravel and sand		26°–30°	100–110 lbs.

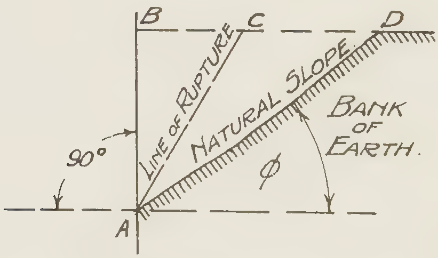


Fig. 253. Angle of repose.

It is found that if the support to a bank of earth be suddenly removed. the earth will usually fracture along a line which is much steeper than the angle of repose. and which is known as the *line of*

rupture; see Fig. 253. It is the smaller wedge of earth ABC which the retaining wall is usually designed to uphold.

The various methods by which the magnitude of the thrust of this wedge of earth on a retaining wall may be estimated are too complex for explanation here, hence we will content ourselves with stating the results of two of them and with showing how they may be applied.

(A) **Earth pressures, Rankine's method.** By this method the pressure at any depth h is given by

$$p = wh\frac{(1 - \sin\phi)}{(1 + \sin\phi)},$$

where w is the weight of a cu. ft. of the material and ϕ is the angle of repose.

The pressure diagram being a triangle, see Fig. 254, the total pressure P, which will act at $\frac{2}{3}$ the depth H, will be

$$P = \text{total pressure} = \frac{wH^2}{2} \cdot \frac{(1 - \sin\phi)}{(1 + \sin\phi)}. \qquad \text{......(ix)}$$

If this expression is compared with expression (viii) it will be seen that pressure from loose granular particles—as here assumed—is less than the pressure which would be exerted by liquid of the same density (see expression viii). The reduction is due to the internal friction which is developed between the particles. Thus the greater the internal friction—that is the larger the angle of friction (ϕ)—the less the side pressure which will be exerted.

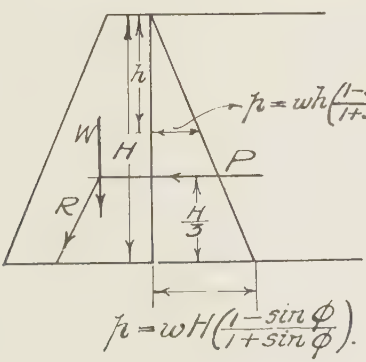


Fig. 254. Rankine's method for finding P.

Having found P, the work of finding the resultant pressure on the base of the wall proceeds exactly as explained in the case of a dam; see Fig. 254.

(B) **Earth pressures—Graphical method.** This method enables us to find P when H, w and ϕ are known. When the surface of the earth is horizontal at the top of the wall the method gives the same results as Rankine's method. It can, however, be used where we have a "surcharged wall", that is when the top surface of the earth slopes upwards from the wall; see Figs. 255 and 256. (**Note.** The same lettering has been used so that the graphical construction in each case should be easy to follow.)

Graphical construction. Let AB be the vertical back of the wall, of height H, which should be drawn to a suitable linear scale.

Draw BD making an angle equal to ϕ with the vertical BA.

Produce CA—the line representing the surface of the earth—to intersect BD in D. Draw the semicircle DEB on DB.

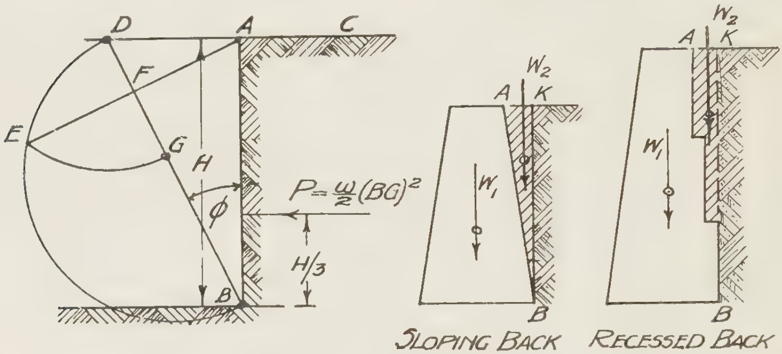


Fig. 255. To find P graphically.

Draw line AFE from A at right angles to BD, cutting the circle in E.

With centre D and radius DE describe an arc cutting DB in G. Measure BG (to the linear scale).

Then P, the total pressure on 1 ft. of the wall, is given by

$$P = w/2\,(BG)^2. \qquad \ldots\ldots(x)^*$$

In cases where the back of the wall is not vertical the same methods may be applied as already described, by treating the wall as having a vertical back, and compounding with the weight of the wall (W_1) the weight of the earth (W_2) included between the back of the wall and the vertical line BK; see small sketches in Fig. 255.

Example. *Find the pressures at the base of the surcharged retaining wall shown in Fig. 256. The weight of the material of the wall is* 150 *lbs. per*

* For a proof of this construction and for a fuller discussion of the various theories applied to retaining walls the reader is referred to Andrews' *Theory and Design of Structures.*

cu. ft., while (w) *the weight of the earth (gravel) may be taken at* 110 *lbs. per cu. ft. and* ϕ *as* 30°.

Graphical construction. The procedure is exactly as described above, the same lettering being used. BG measures 13·4 ft.

Hence $$P = w/2\,(BG)^2 = 110/2 \times 13{\cdot}4^2$$
$$= 9900 \text{ lbs.}$$

P acts at $H/3$ from the base, that is 6 ft. 8 ins.

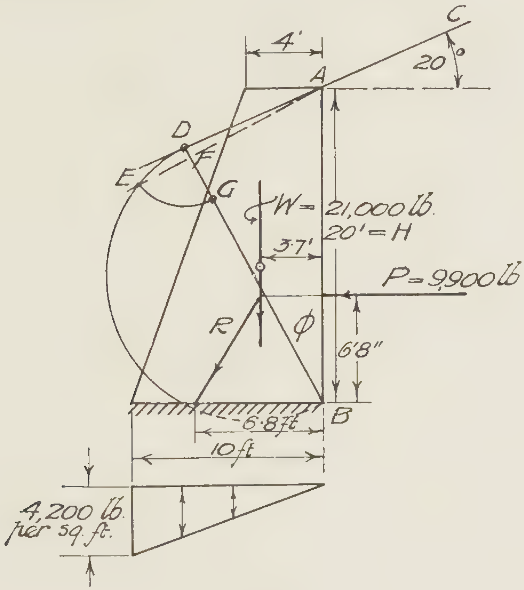


Fig. 256. To find P for a surcharged retaining wall.

The total weight of the wall is given by

$$W = 150\left(\frac{10+4}{2}\right)20 = 21{,}000 \text{ lbs.}$$

By the usual graphical construction for the centroid of a trapezium we find that this (W) acts at a distance of 3·7 ft. from B.

Combining P and W graphically the resultant R will be found to cut the base near the outer middle third point.

Then the maximum pressure at the outer toe of wall will be given by

$$c = \frac{2P}{bd},$$

where P is the vertical component of R, that is 21,000 lbs. Hence

$$c_{\text{max.}} = \frac{2 \times 21{,}000}{1 \times 10}$$
$$= 4200 \text{ lbs. or nearly 2 tons per sq. ft.}$$

The angle which R makes with the vertical is 25°, hence it would be necessary to adopt some measures to prevent sliding, such as carrying the wall well down into firm ground, or "benching" the rock foundation if this be near the surface.

Problems XXI

1. If in the stanchion, with footings and foundation, shown in Fig. 244, the total load to be transmitted to the ground is 100 tons, calculate the minimum area of (*a*) the stanchion base plate, (*b*) the brick pier, and (*c*) the foundation concrete, if the working stresses are as follows: stone 12 tons per sq. ft., brickwork 8 tons per sq. ft., and earth 2 tons per sq. ft.; see Table X.

2. If the foundation block in Prob. 1 be 7 ft. 6 ins. square on plan, calculate the maximum and minimum pressures on the earth if the load of 100 tons does not act centrally but at 6 ins. from the centre of the block.

3. The cross section of a brick chimney shaft is a hollow square of 5 ft. side, the walls being 9 ins. thick. Find the eccentricity of the vertical load which will just reduce the pressure at one edge of a joint to zero. (**Note.** Start by putting expression (iv) equal to zero.) What total load could be carried in this position if the safe stress on the brickwork is 8 tons per sq. ft.?

4. If the chimney shaft in Prob. 3 is built 27 ft. above the ground, of brickwork weighing 130 lbs. per cu. ft., find, by drawing or by calculation, the total wind pressure P (which may be taken to act at half the height of the chimney) which would just reduce to zero the stress on the windward side of the joint at ground level.

5. If the buttress shown in Fig. 250 be increased in thickness to 4 ft., find where the line of pressure (obtained as explained in para. 193 and for the same applied force) cuts each of the joints.

6. Divide the dam shown in Fig. 251 into four parts by drawing joints which are 4 ft. 6 ins. apart vertically, then construct the line of pressure (*a*) when the reservoir dam is empty, and (*b*) when it is full.

7. A retaining wall similar to that shown in Fig. 256 is used to retain a bank consisting of earth which weighs 120 lbs. per cu. ft. and for which the angle of repose is 30°. If the earth is level at the top of the wall, and the dimensions of the wall are: height 16 ft., base 8 ft., top 4 ft., and weight of material of wall 140 lbs. per cu. ft., find the point at which the resultant force cuts the base joint of the wall, the maximum and minimum pressures and the horizontal thrust at this joint.

8. Rankine's theory of earth pressure (on which expression (ix) is based) is sometimes used to find the *minimum* depth to which the foundations of a wall or structure must be carried to prevent the earth being forced up at each side by the pressure of the wall.* The expression which is used is given below. Using this expression, find the minimum depth of the foundations for the stanchion mentioned in Prob. 1.

If d be the minimum depth of the foundations in feet, then

$$d = \frac{W}{wA}\left(\frac{1 - \sin\phi}{1 + \sin\phi}\right)^2,$$

where W = weight of structure = 100 tons in this case,

w = density of the earth = 120 lbs. in this case,

A = area of the base of the foundations = 50 sq. ft. in this case,

and ϕ = angle of repose for the earth = 30° in this case.

* For an elementary explanation of this interesting application of Rankine's theory of earth pressure the reader may consult Andrews' *Theory and Design of Structures*.

CHAPTER XXII

MASONRY OR BLOCK CONSTRUCTION—ARCHES

196. The Masonry Arch. The masonry arch depends for its strength and stability upon the fact that, if properly designed, the separate blocks or voussoirs of which it is composed are so placed in relation to each other, and to the opening which is being spanned, as to be subjected only to compressive forces.

The extended use of the arch through many centuries has led to the adoption of more or less standardised forms and proportions for all relatively small arches, so that the dimensions of such arches in buildings are determined, more often than not, by aesthetic considerations rather than by reason of the structural purpose of the arch itself. It seems, however, a matter of some importance that both those who design and those who construct such arches should have some knowledge of the theory of arch construction. Where the construction of large or unusual arches has to be undertaken, this knowledge is essential and should be as complete as possible.

For certain types of permanent buildings and massive works of construction, the arch is still probably the most effective and graceful method of spanning a large opening. Though it cannot be used over the greater spans now bridged by steel and reinforced concrete structures, the continued use of the arch in suitable circumstances—especially on the Continent—shows that, in spite of its antiquity, it embodies a mode of construction still worthy of careful study.

197. Loaded chain of links.

Experiment. Let a chain of links supported by two spring balances, see Fig. 257 (*a*), be set up as explained in Chap. IV and loaded with a series of loads W_1, W_2, etc. In this case the links should preferably be of equal length and odd in number. Two series should be taken: (*a*) one in which the loads are unsymmetrically arranged about the centre line between A and B, the two points at which balances are attached, and (*b*) one in which the loading is symmetrical and the points A and B lie on the same horizontal line.

In each of these cases the magnitude and direction of each of the forces R_a and R_b should be noted (after making due allowance for the weight of the apparatus), together with the direction and position of each of the links. These values should then be used to check the following statement.

Case (a). In the case of the unsymmetrical loading, if the links be looked upon as forming a link polygon, see Chap. IV, a polar diagram may be drawn which, for the given values of W_1, W_2, etc., will give the magnitude and direction of the reactions R_a and R_b. It should be noted

that if, in the experiment, the spring balances at A and B are attached to the ends of the last links, then the centre line of the balance and the link must, in each case, lie in one and the same straight line. The lines radiating from p in the polar diagram will also give the forces acting in the links.

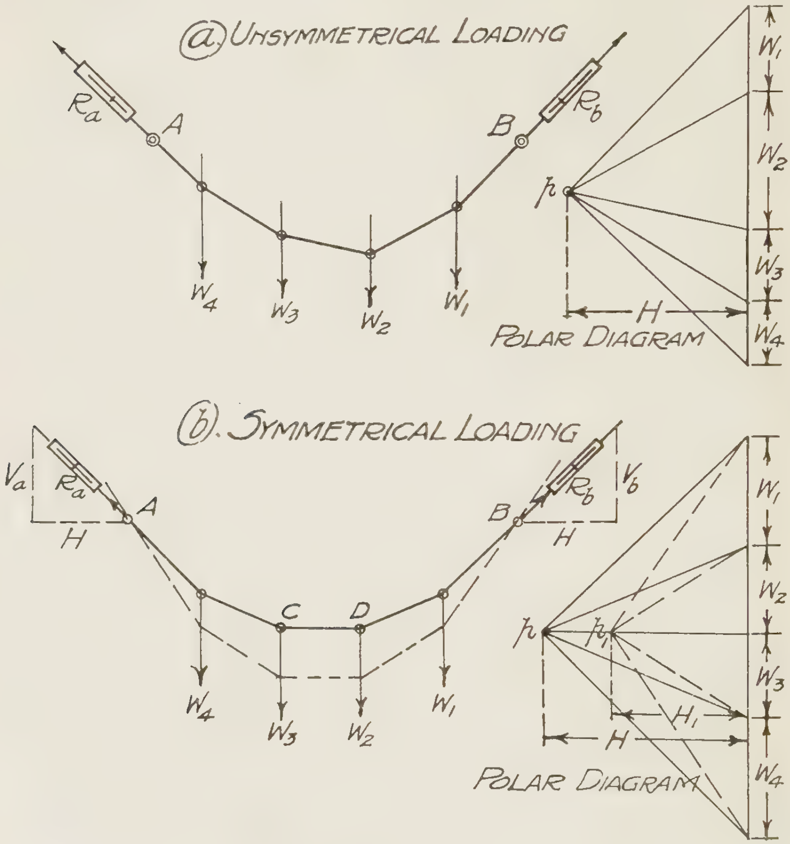


Fig. 257. Chain of loaded links.

If now H, the horizontal distance between p and the "load line", be measured on the force scale, this will obviously give the horizontal components of R_a, R_b and of each of the forces acting in the links. For the present we will call H the "horizontal pull" in the chain of links.

Case (b). Exactly similar results will be obtained in the second case, but, in addition, since the loading is symmetrical and the points A and B on the same level, the middle link CD will also be level and the force acting in it will be equal to H, the horizontal pull in the chain, this force being also equal to the *horizontal component of each of the reactions*; see Fig. 257 (*b*).

It will be useful to note, in this case, that, by increasing the lengths of the links proportionately, another outline of the chain might be obtained, see dotted figure, which, for the same loads, would have a different horizontal pull (H_1); in the case shown H_1 would be less than the first value H.

(**Note.** Such an arrangement of links and loads gives, in a simple form, the principle which underlies the construction of suspension bridges.)

198. The linear arch. In the cases dealt with in the last paragraph the links were in tension. If now we reverse the position of the links, the other conditions remaining the same, see Fig. 258,

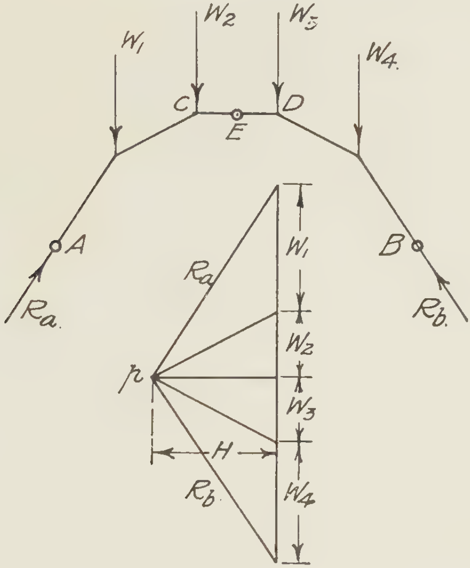


Fig. 258. The linear arch.

the links will obviously be in compression under the action of the loads. As before, the magnitudes of the forces acting in the links, and also the magnitudes of the reactions, will be given by the polar diagram. Such an arrangement of links, from its similarity to the arch, is usually known as a **linear arch.**

A brief consideration will show that such an arrangement of links would be in *unstable equilibrium*, since the slightest movement of either links or loads would cause the whole to collapse. The case can, however, be considered experimentally where there are only two links.

Experiment. Set up a piece of apparatus as shown in Fig. 259 (this corresponds to the ordinary experimental roof truss). Attach two equal

loads W_a and W_b at points equidistant from the apex E. (The loads W_a and W_b may be taken to be the resultant of several loads on each side of E, while the point E is assumed to be taken in this case at the centre of the middle link CD shown in Fig. 258.)

The reactions at A and B cannot be measured directly, but if a spring balance be inserted between A and B this will give H, the horizontal component of each reaction. (The vertical components, also equal, may be measured experimentally by supporting A or B on the table of a small weighing machine.) Make a record of the magnitudes of W_a and W_b and also of the size and position of each link.

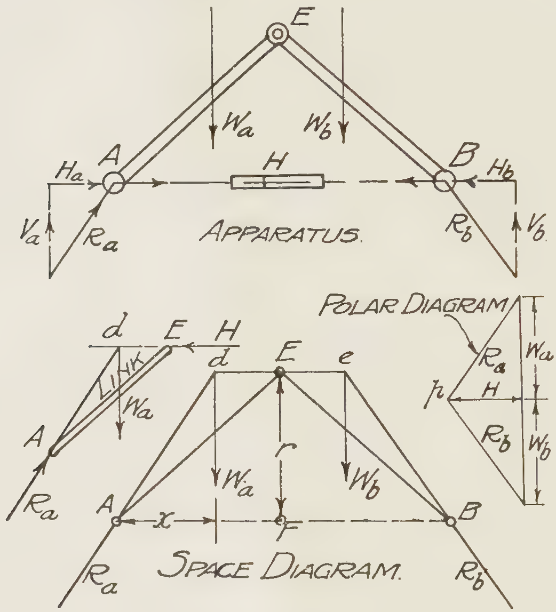


Fig. 259. Experimental linear arch of two links.

Now set out the apparatus to scale, as in the space diagram in Fig. 259, and it will be clear that, from the information already obtained, the lines of action of R_a and R_b can be drawn. Consider the equilibrium of the link AE, see small sketch; since it is held in equilibrium by the action of three forces, they must be concurrent and their lines of action will intersect in some point d.

To find d produce the line of action of R_a to cut the line of action of W_a in d. The point e is found similarly. Then dE and Ee will be the lines of action of the reactions at E, the upper end of each link; but since the loading is symmetrical the reactions at E must both be horizontal, hence, if the work has been done accurately, dE and Ee should form a single horizontal line.

Further, the figure $AdEeB$ will form a link polygon for the two loads W_a and W_b. The horizontal distance H from p to the load line on the polar diagram (constructed in the usual way) will give the magnitude

of the reactions at E, as well as the horizontal components of R_a and R_b, which are equal. All these statements should be checked by means of the values obtained experimentally.

To find H by calculation. Since the links are in compression, then H will be the *horizontal thrust in the arch.* The value of H may be found by calculation. If the height r be called the "rise of the arch", and x be the horizontal distance from A to the line of action of W_a, then by taking moments about A we have

$$H \times r = W_a \times x,$$

or

$$\boldsymbol{H} = \textbf{horizontal thrust} = \frac{\boldsymbol{W_a \times x}}{r}. \qquad \text{......(i)}$$

From expression (i) it is easy to see that $\boldsymbol{H}$ *will vary inversely as the rise of the arch, so that, for the same span and loading, the greater the rise the less the thrust and* vice versa.

199. The theory of arch construction. The generally accepted theory of arch construction is based upon the idea of a *linear arch contained within the outline of the arch structure,* as shown in Fig. 260, in which $AFCDB$ is a linear arch drawn for the given loads, R_a and R_b being the reactions at the abutments.

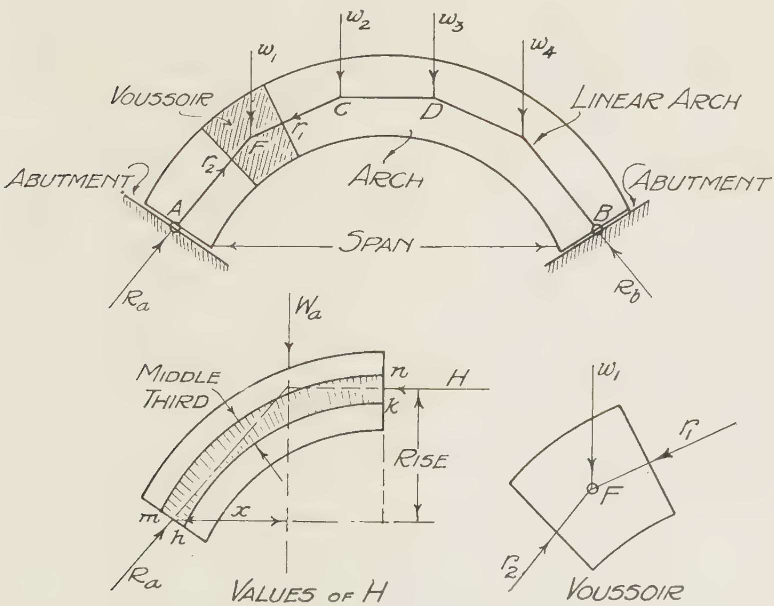


Fig. 260. Finding H for a symmetrically loaded arch.

Let us now consider the equilibrium of a single voussoir, which for convenience we will assume to be massed about the load point F. The voussoir, see small sketch, is evidently held in equilibrium by the action of three forces, viz. w_1, the load coming on the voussoir (which we can assume includes the weight of the voussoir), and the two forces r_1 and r_2 which correspond to the forces in the links of the linear arch at this point.

If we now apply the three masonry rules set out in the preceding Chapter, we shall arrive at a corresponding set of conditions which the masonry arch must satisfy. These conditions are:

I. To avoid tension in any of the arch joints, the forces acting in the links of the linear arch—that is in the arch itself—must lie within the middle third.

II. The stresses induced at each joint by the forces acting in the links of the linear arch must not exceed the safe compressive stress for the material.

III. To avoid slipping at the joints, the angle which the forces acting in the links makes with the normals to the joints must in no case exceed the angle of friction (ϕ).

The values of H. For the condition of no-tension (I) it is clear that the linear arch must be drawn within the limits of the middle third. Consider the drawing, Fig. 260, showing half a symmetrically loaded arch, where W_a is taken to represent the resultant of all the loads on this portion of the arch. Quite a range of values of H may be found which satisfy condition I at the middle joint and the abutment, since H may act at any point between n and k, while R_a, the reaction at the abutment, may act at any point between m and h. It will be convenient to consider only the two extreme limits, that is the one which gives us (a) the *minimum value of* H, when H is as high as possible and acts at n while R_a acts at h; and also (b) the *maximum value of* H, when H is as low as possible and acts at k, while R_a acts at m.

The problem then resolves itself into drawing a linear arch or, as it is also called, a "line of pressure", within the middle third of the arch, for which the horizontal thrust has a value lying between the two extremes mentioned above. Then it may be shown (though the proof is beyond the limits laid down for this volume) that, if such a line can be drawn, the arch will be stable, provided that, in addition, the conditions II and III as to safe pressure and sliding at the joints are satisfied.

200. Arch loading. An arch usually carries, in addition to its own weight, some part of the weight of the structure built above it. The total load thus occasioned may be made up of two parts,

(*a*) a *dead or permanent load*, which does not alter, and (*b*) a *live or movable load*, which may vary in amount or in position. In the majority of cases the live or movable load is not great compared with the dead load plus the weight of the arch itself. This is particularly true of arches in buildings, as distinct from arches built to carry moving traffic loads. It is therefore usually sufficient to allow for the effect of live load by increasing the magnitude of the dead load by a suitable amount, the stability of the arch being checked first under (*a*) the constant dead load, and second under (*b*) the dead load plus the live load allowance.

In those cases where the live load may occur over only a portion of the span, an unsymmetrical form of loading will be produced, and, since *it is unsymmetrical loading which usually produces the greatest extremes of stress in an arch*, care must be taken to see that the arch is checked for those cases in which the loading is the most markedly unsymmetrical. There are therefore two cases with which we shall deal and into which all ordinary cases fall; they are (A) arches with symmetrical loading, and (B) arches with unsymmetrical loading.

Equivalent load of uniform density. The loading on an arch may be made up of pieces of construction, of varying thicknesses and densities. It will be found to simplify the working considerably if this varied loading is reduced to an equivalent loading of the same density and thickness as the arch ring—or portion of the arch ring if only a "slice" of it is being considered; see the example given in para. 201.

Having done this, we may ignore the joints of the arch ring and divide up the total equivalent loading—including the weight of the arch ring—into vertical strips of convenient width. If the weights of these strips are calculated, they may be taken to act at their C.G.'s or (approximately) at the centre of the width of each strip, so forming the weights w_1, w_2, etc. for which the linear arch will be drawn. The procedure will be readily followed from the example given in the next paragraph.

201. Example. *To ascertain the stresses in an arch with symmetrical loading.*

Particulars are given in Fig. 261 of a brick arch of 20 ft. span carrying an 18 in. brick wall above it, which remains the same thickness as the arch ring to a height of 1 ft. above the crown. At this level the wall is reduced to 14 ins. in thickness and, on the offset so formed, rests a floor. The loading from this floor may be taken to be 600 lbs. per lin. ft. In the 14 in. wall above, a window opening occurs as shown. The considerations which help us to decide what load is being carried by such an arch are discussed in para. 205; in the present case we will assume that the arch carries the whole of the brickwork included in the figure *ABFGDE*. The brickwork may be taken to weigh 120 lbs. per cu. ft.

Equivalent load of uniform density.

Floor load. This may be expressed, above the line BB, in terms of brickwork 18 ins. thick rising to the line $B'B'$, see Loading Diagram, Fig. 261 A. A load of 600 lbs. per lin. ft. represents (600/120) or 5 cu. ft. of brickwork. At 18 ins. in thickness the height of this ($B'B'$) will be (5 × 12/18) or 3 ft. 4 ins.

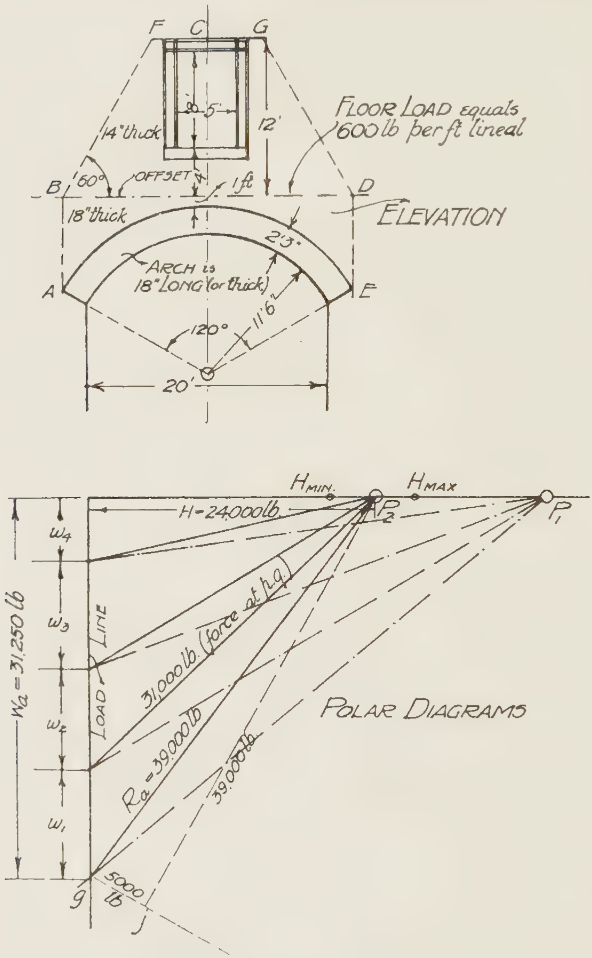


Fig. 261. Arch with symmetrical loading.

The height of the equivalent figure $B'F'G'$, etc. will evidently be in the ratio (14/18) to the height of the original figure BFG, etc., hence the height will be (12 × 14/18) or 9 ft. 4 ins.

We can now set out the load figure $ABB'F'G'HK$ for half the arch, since the loading is symmetrical; see Fig. 261 A. This can be divided into strips of convenient width as shown, the strips in this case not being

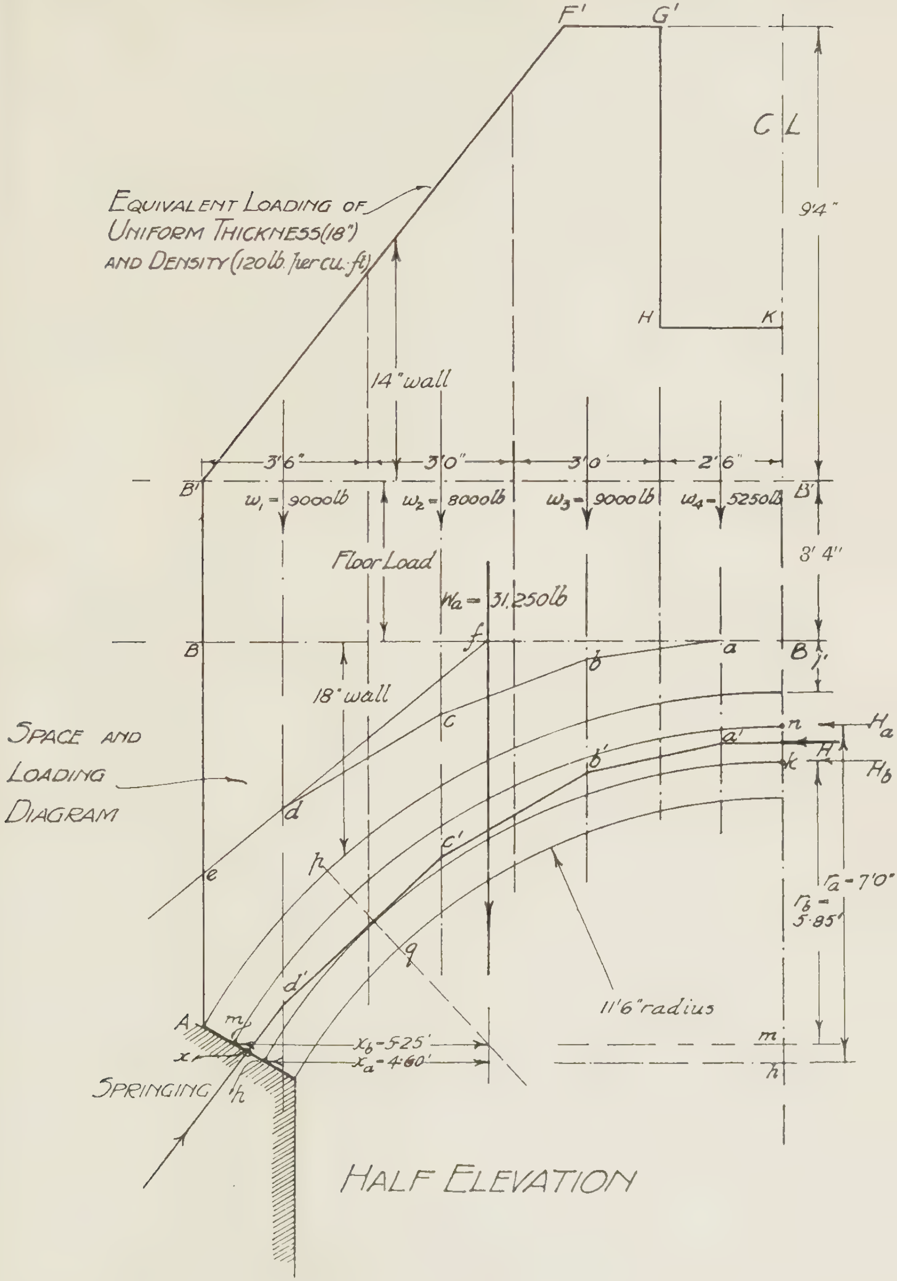


Fig. 261 A. Space and loading diagrams.

equal. The volumes of these strips have been taken to be approximately proportional to the lengths of the centre lines of the figures from which, by calculations which need not be given here, the weights w_1, w_2, etc., have been found. The magnitudes of these loads, which are taken to act down the centre lines of the figures, are indicated in Fig. 261 A.

To find the line of action of W_a, the resultant of all the loads on this half of the arch. A load line is drawn to a suitable scale from which is obtained the magnitude of W_a, viz., 31,250 lbs. Drawing a horizontal line from the upper end of the load line, see Fig. 261, and taking any suitable pole P_1, a polar diagram is drawn. With the aid of this a link polygon *abcdef* is drawn, preferably above and away from the arch, see Fig. 261 A. Evidently the intersection of the first and the last lines at f will give a point on the line of action of W_a; see Chap. IV.

To find the maximum and minimum values of H. The distances x and r have each two extreme values, these are indicated in Fig. 261 A. Then from expression (i),

$$H = \frac{W_a \times x}{r},$$

whence

$$H_{\text{max.}} = \frac{31{,}250 \times 5{\cdot}25}{5{\cdot}85} = 28{,}000 \text{ lbs.},$$

$$H_{\text{min.}} = \frac{31{,}250 \times 4{\cdot}6}{7} = 20{,}500 \text{ lbs.}$$

Both these values are marked on the horizontal line through P.

To find a suitable line of pressure. As a first trial we will take P_2 about half-way between the two extreme values, say $H = 24{,}000$ lbs., and commence the first line of the link polygon, which is horizontal and represents the line of action of H, half-way between the points n and k, where the middle third lines cut the centre line of the arch. The drawing of the link polygon, or line of pressure, $a'b'c'd'$ should not occasion any difficulty. This line lies wholly within the middle third, hence a further trial is unnecessary. The next step is to check the pressures induced at what appear to be the most highly stressed joints, including always the joints at the crown and the springing.

Pressures at the crown joint. The force H is normal to the joint and acts at its centre, therefore

$$\text{Stress} = \frac{24{,}000}{27 \times 18} = 50 \text{ lbs. per sq. in.}$$

Pressures at the springing. The line of pressure crosses the joint practically at its centre; it is, however, inclined to the line of the joint and we must therefore resolve it into its components normal and parallel to the joint. This is readily done by drawing lines parallel to these directions on the polar diagram; see triangle $P_2.g.j$. From this we see that the normal pressure is 39,000 lbs. Then

$$\text{Stress at springing} = \frac{39{,}000}{27 \times 18} = 80 \text{ lbs. per sq. in.}$$

Pressure at worst joint. This will usually be the joint at which the line of pressure approaches nearest to the middle third line. In this case the closest approach occurs where the joint pq has been drawn. The line of pressure is so very near to the middle third line that we shall assume that it actually coincides with it.

The maximum pressure at joint pq will occur at the edge q, and is given by $c_{max.} = 2P/bd$; see (iii), Chap. XXI.

In this case the line of pressure is practically normal to the joint and the force equals 31,000 lbs.; see polar diagram, Fig. 261. Therefore

$$\text{Maximum stress at } pq = \text{twice the mean stress} = \frac{2 \times 31{,}000}{27 \times 18}$$

$$= 128 \text{ lbs. per sq. in.}$$

All these stresses will be seen to be well within the safe values given in Table X, and it would appear in fact that, for the given loading, the dimensions of the arch, say the depth, might be safely reduced.

202. Arches with unsymmetrical loading —The three-pinned arch. The chief difficulty to be met with in dealing with cases of unsymmetrical loading, is that involved in finding the direction of the reactions at the crown of the arch—and of course, correspondingly, at any other joint in the arch. If this direction can be settled, then the work necessary to complete the solution is little more difficult than, and follows exactly the same lines as, in the case already treated. A somewhat elaborate method of solution is given in para. 204.

In many cases a solution can be obtained readily by assuming that the arch is supported on three pinned joints, at the crown and at each springing, in which case the lines of pressures must pass through these pins and the position of the linear arch is thus fixed. This is the principle of construction of what are known as "Three-pinned arches."

The three-pinned arch. Although this method of constructing an arch is only used as a rule in large arches of dimensions far beyond those which can be considered in this volume, yet the principle of construction affords so excellent an example of the application of fundamental statical laws that a note may be usefully added here. In recent years quite a number of arches, bridges and large roof trusses have been constructed on this principle, the materials used being brick, stone, concrete, reinforced concrete or steel for the main members, and usually cast iron or steel for the pinned joints.

In the case of arches the arch is divided into two convenient portions between which the pinned joints are placed. Full details of the latter need not be given here but, as shown in the sketch in Fig. 262, they usually consist of a smooth pin on which rest two steel or cast iron seatings, from which the masonry or concrete arch is built up or, in the case of steel arches, to which the steel portion is secured. It should be clear to the reader without further explanation that, whatever combination of loading is placed upon such an arch, *the line of pressure must pass through the three pins.*

There is the additional practical advantage in this form of construction, that the arch can accommodate itself to changes in temperature which, in rigidly built arches, is sometimes responsible for considerable additional stresses.

Experiment. Using the same apparatus as that shown in Fig. 259 we may carry out an experiment on what is in effect an unsymmetrically loaded three-pinned arch, by making the two loads W_1 and W_2 unlike (or by placing them unsymmetrically); see Fig. 262.

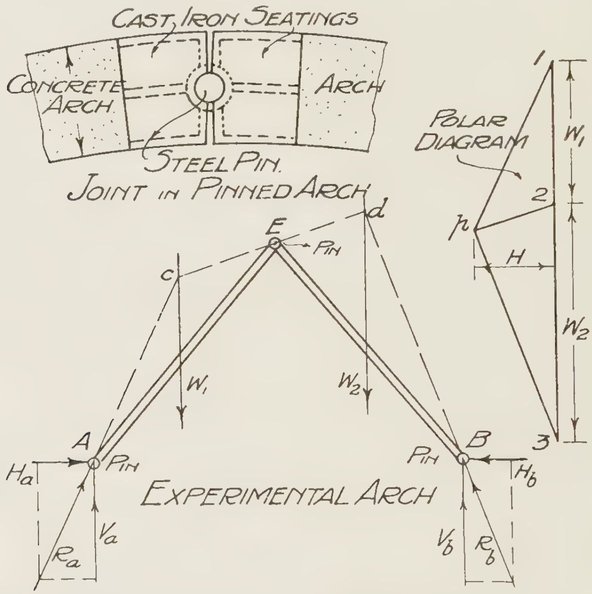


Fig. 262. Three-pinned arches.

In the manner already described we may find experimentally the lines of action of the two reactions R_a and R_b. If these are produced to intersect the lines along which W_1 and W_2 act, in points c and d respectively, then, as before, the three points c, E and D should lie on one and the same straight line. In the present case this will be a sloping line.

If the polar diagram is now drawn, H, the horizontal distance between p and the load line, will be seen to be the horizontal component of the thrust at the centre pin, while $p2$, parallel to cd, will give the magnitude and direction of this thrust.

In practice we must find the reactions at A and B and the thrust at the centre pin E in some other way. We do this by drawing a link polygon to pass through the three points A, E and B, after which, by drawing the corresponding polar diagram, the necessary data are obtained. The method is perhaps best explained by means of an example.

203. Example. *Three-pinned arch with unsymmetrical loading.* To avoid unnecessary calculations we will assume that the arch described in para. 201 is now to be constructed with three pin joints, one at each springing and one at the crown; see A, E and Y, Fig. 263. We will further assume that the loading remains the same except that a large floor beam is to be added at Q, a point 2 ft. 6 ins. to the right of the centre line. The additional load w_5 thus brought into play is 10,000 lbs., the loading being thus made unsymmetrical. We desire to find the direction and magnitude of the thrust at each of the pins.

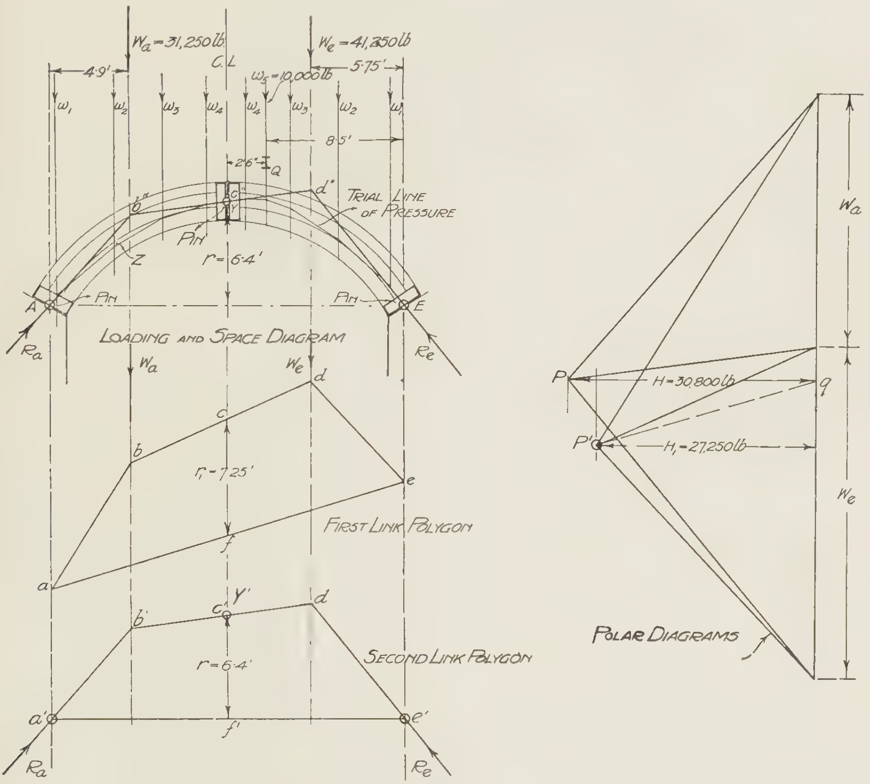


Fig. 263. Three-pinned arch with unsymmetrical loading.

To find the line of action of W_e. W_e is the resultant of all the loads acting to the right of the centre, and corresponds to W_a on the left.

The magnitude of $W_e = 31{,}250 + 10{,}000 = 41{,}250$ lbs.

Now if W_e' be the resultant of all the loads on the right-hand side *except* the new load, we know that its magnitude is 31,250 lbs. and also that it (W_e') acts vertically 4·9 ft. from the vertical through the pin at E (these magnitudes are repeated from the left of the figure, where the various loads and dimensions remain unchanged). By

measurement we shall find that the load of 10,000 lbs. acts at 8·5 ft. horizontally from E; hence, taking moments about the pin E we have, if x is the distance to the line of action of the resultant (W_e),

$$W_e \times x = (10{,}000 \times 8\cdot5) + (31{,}250 \times 4\cdot9),$$

whence $x = 5\cdot75$ ft.; see Fig. 263.

To draw a link polygon to pass through the three points A, Y and E. In a position clear of the elevation of the arch draw any link polygon *abcdef* for the two loads W_a and W_e. On the corresponding polar diagram draw the line $P'q$, parallel to the "closing line" *afe*, which gives the point q, and divides the load line into two parts showing the vertical components of the reactions at A and E respectively.

To draw a link polygon with a horizontal closing line—which is necessary because A and E are on the same level—the pole P in the new polar diagram must lie on a horizontal line through q.

The height *cf* at the centre of the link polygon which we have just drawn is 7·25 ft., see Fig. 263, while the height to the pin Y at the same point is only 6·4 ft. Clearly the height of the polygon must be reduced to the latter height. But we have already seen that H, the horizontal thrust, varies *inversely* as the rise (r), see para. 198. In this case r corresponds to the height of the polygon. Hence, if H_1 and H are the original and the required thrusts respectively, we have

$$\frac{H}{H_1} = \frac{r_1}{r} = \frac{7\cdot25}{6\cdot4} \quad \text{or} \quad H = \frac{7\cdot25 \times 27{,}250}{6\cdot4}$$
$$= 30{,}800 \text{ lbs.}$$

If the link polygon *a′b′c′d′e′f′* is now drawn on a horizontal base *a′e′*, its height *c′f′* at the centre will be found to be 6·4 ft. It can thus be drawn to pass through the three pins A, Y and E, as shown on the drawing of the arch, and the direction and magnitude of the thrust at each pin can be measured on the polar diagram.

204. Masonry arch with unsymmetrical loading. Two simple methods are available by which we may draw the line of pressure in such a case, in the first we make use of the principle of the three-pinned arch and in the second we use a geometrical method attributed to Prof. Fuller.

Example. *For convenience we will deal with the same arch as was illustrated in Fig.* 261, *together with the additional unsymmetrical load added in the Example given in para.* 203.

(A) **First method—utilising the principle of the three-pinned arch.** In this method we assume that three pins are inserted at three convenient points in the arch ring, thus enabling a linear arch to be drawn for the given loading. Having in this way obtained the value of H, the horizontal thrust, we next ascertain whether the complete linear arch, drawn through the three selected points, lies wholly within the middle third of the arch.

The outline of the arch and the middle third lines have, for convenience, been shown on Fig. 263. Using the link polygon *Ab″c″d″E*, to give the position of P on the polar diagram, as well as the value of H, the complete link polygon or linear arch can thus be made to pass through the points A, Y and E.

The complete line of pressure in this case passes outside the middle third at Z on the left, see Fig. 263. The next step is therefore to ascertain *by trial* whether, if the relative positions of the three points A, Y and E are altered, a new pressure line can be drawn which will be entirely within the middle third. In some cases a solution can be quickly arrived at in this way, but in others the method may be long and tedious, in which case the second method should be adopted.

(B) **Second method—Prof. Fuller's method.** The drawing of the arch and loading has been repeated in Fig. 264 (i), while each successive step is shown on a separate diagram for clearness.

(a) The position of q, which divides the load line of the polar diagram into two parts, corresponding to the vertical components of the reactions at the springings, is found as already explained. Any convenient polar point P_1 is then taken on a horizontal line through q and, with the aid of the polar diagram so formed, the link polygon $abcdefghkl$ is drawn, commencing at the centre point of the springing a and finishing at the corresponding point m on the other side.

From the highest point f two inclined straight lines fn and fo are now drawn, the points n and o being placed anywhere on the line am produced.

(b) *Each vertical* through the angular points a, b, c, etc. of the link polygon is now treated in the same manner as described below for the vertical through d.

A horizontal line through d is drawn to cut the line fn in d'. A dotted vertical is now drawn from d'. On this new vertical two points r' and s' are marked which are horizontally opposite to the points r and s, in which the first vertical through d cuts the middle third lines. *This process is repeated for each vertical.*

(c) The dotted verticals, such as $d'r's'$, are now transferred to a new base line $n'o'$. Through the points so obtained a distorted figure, corresponding to the middle third portion of the arch, is drawn; see figure $t''v''w''u''$ in Fig. 264 (ii).

(d) If possible draw *within this figure* two straight lines which intersect on the vertical through f' and have *the steepest possible inclination*; see the lines $b''e''$ and $h''l''$ in Fig. 264 (ii).

Now by our construction the distorted figure *$t''v''w''u''$ has the same degree of horizontal distortion as had the link polygon* abcd, *etc. when compared to the straight lines* nf *and* fo. Hence if the distorted figure is restored to its original shape, while the points c'', k'', etc. on the straight lines $b''e''$ and $h''l''$ are kept in the same vertical relations to it, then we shall have the outline of a link polygon lying wholly within the middle third figure.

(e) The outline of the arch and the middle third figure has been repeated for clearness in Fig. 264 (iii). On this figure the solid vertical lines correspond to the original load lines of the arch. The points b''', c''', d''', etc. are now transferred to these lines. When these are joined we have the required link polygon.

(f) From this link polygon a new polar diagram is drawn which shows that H_2, the horizontal thrust in the arch, is 30,500 lbs.

The stresses at the joints may be found in the manner described in para. 201, thus near the point f''', where the line of pressure coincides with the edge of the middle third, the total pressure is practically normal to the joint and equal to 30,600 lbs., whence

$$\text{Maximum stress} = \frac{2 \times 30{,}600}{27 \times 18} = 126 \text{ lbs. per sq. in.}$$

Note. If it had not been possible to draw the two straight lines within the distorted figure of the middle third in Fig. 264 (ii), then the arch would have had to be re-designed and its depth increased or its shape altered.

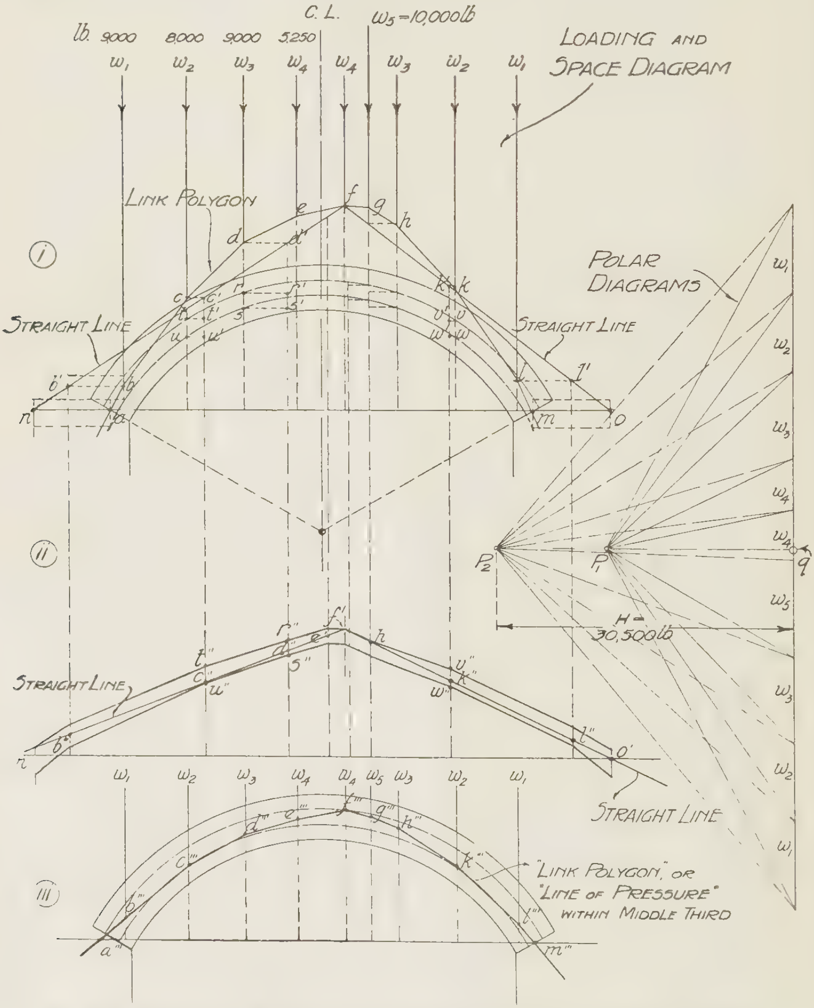


Fig. 264. Masonry arch with unsymmetrical loading—Prof. Fuller's method.

205. Additional notes on arches.

(a) **Loads on arches.** Where openings occur in lengthy walls of brick or stone construction, it is usual to assume that the arches (or lintels) spanning such openings carry only that portion of the wall above which

is included within an isosceles triangle, see triangle ABC in Fig. 265 (i), with a base equal to the width of the opening or arch and having base angles of 45° or 60° according to the character of the wall, the larger angle being taken in the case of the less well-constructed walls.

This assumption is based upon the idea that, if the arch or lintel were removed, there would be a corbelling-out of the masonry as suggested on the right, the material falling away below the stepped line. Obviously this effect would not occur in those cases where the opening is (*a*)

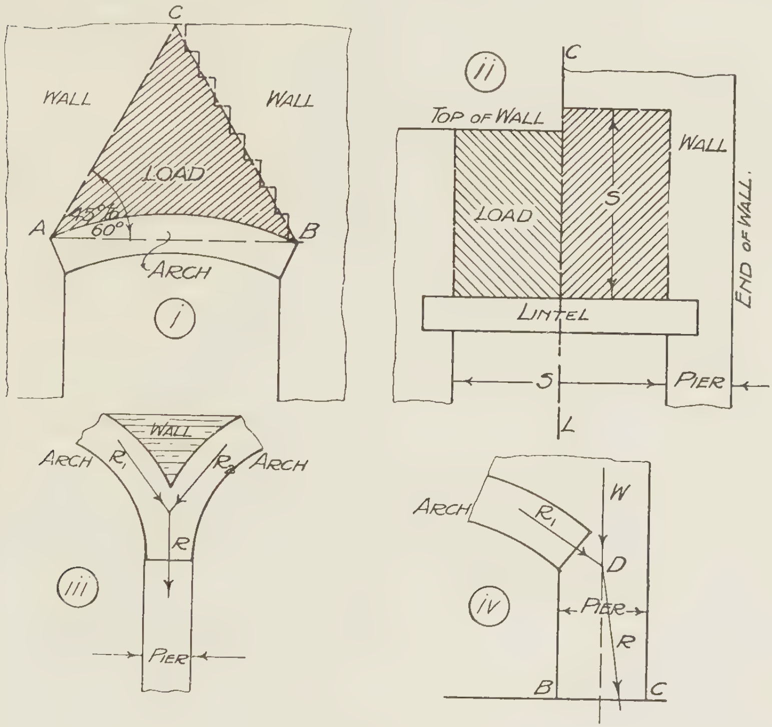


Fig. 265. Loads on arches.

near the top of the wall, or (*b*) near the end of the wall; see Fig. 265 (ii). In both these cases the load, taken over the whole span, is assumed to include the wall up to the top (and any load that may rest thereon), or at least to a height equal to the span. In less simple cases the matter is one to be settled by judgment and experience.

(b) Forces at the abutments. If in a series of equal arches the reactions R_1 and R_2, see Fig. 265 (iii), be equal in magnitude and equally inclined, then the piers between the abutments may be relatively slender. (Good examples of this can be seen in some of the Gothic churches, where not only are the reactions balanced, but the loads carried by the arches are lessened by reason of the shape of the arches, which approaches the 60° isosceles triangle to which we have referred.)

If, however, an arch, either alone or as the last in a series, occurs near the end of a wall, see Fig. 265 (iv), then it is clear that the width of the pier must be such that R (the resultant of the reaction R_1 of the arch and W the weight coming on to the pier from the building above) falls within the middle third of the joint at BC as already explained.

(c) **Depth of arches.** Having ascertained either H, the horizontal thrust of an arch, or one or other of the reactions from a consideration of the total load coming upon it, then it is possible to settle the dimensions of the arch ring from a consideration of the safe stresses. In large arches it is common practice, however, to find the trial depth of the arch ring in the first place by means of a formula.

Trautwine's is a good example of such rules and gives

$$d = \text{depth of arch at crown} = \frac{\sqrt{r + \frac{1}{2}S}}{4} + 0{\cdot}2, \qquad \ldots\ldots\text{(ii)}$$

where S is the span and r the radius of the arch at the crown (all in feet). For brickwork the size may be increased by one-third.

For ordinary brick arches one half-brick ring may be adopted for each five feet of span.

It should be noted that these rules do not apply to very flat arches or to arches in which the rise is considerable in proportion to the span. They apply very well to arches which include an angle of about 120° at the centre; see Fig. 261.

(d) **Failure of arches.** It is sometimes stated that an arch will fail if the line of pressure passes outside the middle third. As we have seen, however, in the case of masonry walls, this need not necessarily be so. *If the compressive pressures induced are not too great*, then it may be possible occasionally to increase to the "middle half" the limits within which the line of pressure must lie. In such a case there would of course be tension in the joints. Much will depend upon the magnitude of the stresses induced and also, in the case of arches, upon the nature of the work surrounding the arch. For example, in deep arches of the Gothic or semicircular type, the line of pressure may and usually does pass well outside the middle third near the base of the arch. But failure is not likely to take place if, as shown in Fig. 265 (iii) and (iv), there is adequate weight of solid material behind the arch to resist this thrust. Each case must receive careful consideration, both as to the maximum stresses induced, and also as to the tendency to slip in the manner explained in this and the preceding chapter.

Experiment. An experimental arch may be constructed with wood voussoirs having fairly thick sheets of soft rubber inserted at the joints. When the arch is loaded, the position of the line of pressure can be traced by noting the varying strain on the rubber joints.*

The supporting forces may be obtained experimentally by mounting the arch on rollers, supporting one end on a table-weighing machine, and restraining the horizontal movement of the supports by means of an adjustable tie in which a spring balance is inserted (see also experiments described in paragraphs 198 and 202). By means of this adjustable tie it is possible to maintain the arch at a constant span during the experiment and also to compensate for the extension of the balance.

* An experimental arch of this type was described in a paper on "Masonry and concrete arches," read by Mr A. A. Fordham to the Institution of Structural Engineers, and reported in *The Structural Engineer*, June, 1929.

Problems XXII

1. If in Fig. 257 (*b*) the weights W_1, W_2, W_3 and W_4 are each 10 lbs. and 6 ins. apart horizontally (so that the distance AB is 30 ins.), find *by calculation* the value of H when the link CD is 6 ins. below the line joining A and B. What would the distance between the lines AB and CD have to be to reduce H to 20 lbs.?

2. Using the methods described in para. 201, draw a suitable line of pressure for the arch in Fig. 261 when subjected only to the weight of the arch ring itself. Give the position of the points at which this line of pressure crosses the joints at the crown and the abutment.

3. Draw a suitable line of pressure for the arch in Fig. 261 when carrying a uniform wall equal in thickness to the arch ring and reaching up to a level line BD which is 2 ft. 6 ins. above the crown of the arch. Give the position of this line of pressure at the crown and abutment joints, and state where it approaches most nearly to the middle third lines.

4. Find by calculation (and check by graphical methods) the values of the components H, V_a and V_b in the experimental arch shown in Fig. 262, if AB is 32 ins. and AE and BE are both inclined at 45° to the horizontal, the loads W_1 and W_2 acting at the middle points of these members and being respectively 14 lbs. and 21 lbs.

5. Draw a suitable line of pressure for the symmetrically loaded arch (only one-half need be drawn) shown in Fig. 261, using Prof. Fuller's method.

6. The arch shown in Fig. 261 finishes near the end of a wall as in Fig. 265 (iv), the pier being the same thickness as the arch ring (18 ins.). The width of the pier is 9 ft. and the vertical load W, due to the weight of the upper structure of the building, acts down the centre line of the pier. The thrust from the arch is 40,000 lbs., acting at an inclination of 52°, and it intersects the centre line of the pier at the point D, 10 ft. above BC. Find the minimum value of W so that the resultant force R just crosses the joint BC on the middle third line. What is the maximum stress on this joint under these conditions?

CHAPTER XXIII

STRENGTH IN COMPRESSION. COMPRESSION MEMBERS AND COLUMNS IN STEEL AND CAST IRON

206. Compressive strength of structural materials. It is convenient when dealing with the compressive strength of structural materials to divide them into two groups, viz. (A) ductile materials, and (B) brittle materials. There are a few materials which exhibit some characteristics belonging to both groups but, in the main, the division holds good and is a useful one to make. For reasons to be made clear presently we shall consider in this paragraph only the strengths of very short specimens, in which the height does not greatly exceed the diameter or least lateral dimension.

(A) **Ductile materials.** Under this head may be mentioned mild steel, wrought iron, aluminium and lead, but we shall only concern ourselves here with mild steel.

Mild steel. The stress-strain curve for mild steel in compression is very similar to that for tension, see Chap. XII, though the elastic limit and the yield point are usually less well defined. Beyond the yield point the material becomes more and more *plastic* so that, after a certain stress is reached, the material continues to "flow" almost indefinitely. Thus, if a cylindrical piece of mild steel be tested in compression, it will, after a time, be reduced in height and increased in its lateral dimensions, as shown in Fig. 266.*

Lead is a well-known example of a material which becomes plastic under a relatively low pressure (about 1000 lbs. per sq. in.), thus enabling it to be "worked" by pressure or by "beating".

In the harder steels the final failure will be somewhat similar to that for brittle materials, and will thus be quite definite. In the case of mild steels, however, we are compelled to adopt some sort of convention as to when failure occurs. It has been suggested that the stress when the "plastic state" is reached should be taken as the ultimate compressive stress.

(B) **Brittle materials.** In the brittle materials there may be only a slight increase in the lateral dimensions of the specimen and failure will usually occur quite suddenly, by fracture along planes which are more or less steeply inclined to the base of the specimen; see Fig. 266.

* For complete details of a compression test of mild steel the reader may consult Prof. C. A. M. Smith's *Testing of Materials.*

It may be shown that this is the form of failure which is likely to occur in all such materials, since the compressive stress tends to induce a tensile stress across the specimen; see para. 135. Under the combined action of two such stresses there will be set up shear stresses on planes inclined to the first two stresses. In para. 135 it was shown that, if the original compressive and tensile stresses were equal and acted at right angles to each other, then shear stresses of the same intensity would act along planes inclined at 45° to the first planes. As we shall find by actual tests the fracture

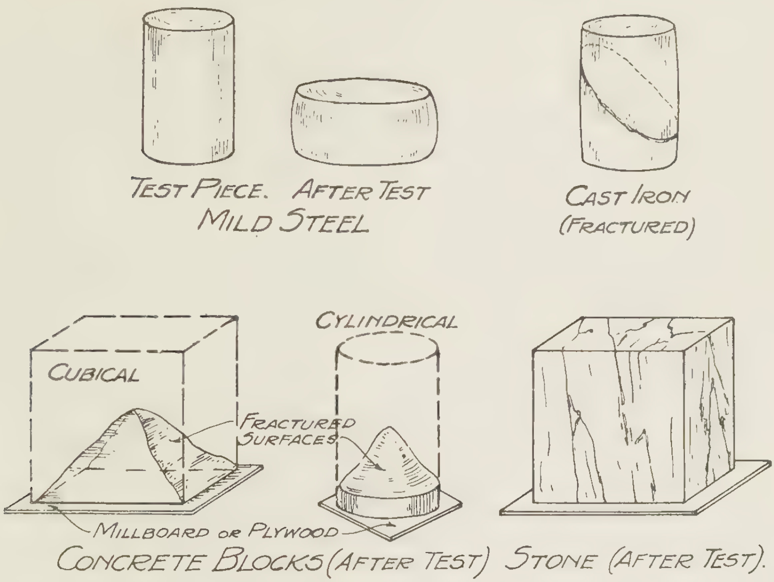


Fig. 266. Compression tests.

planes are usually more steeply inclined than 45°, but it would be out of place to attempt an explanation in this volume, and we may therefore assume that this form of fracture is due to the shear stresses set up in the specimen.

Cast iron. The character of the fracture in this brittle material is sufficiently indicated in Fig. 266. It will usually be accompanied by a very slight lateral swelling of the specimen.

Stone and brick. The planes of fracture in these materials are usually very steeply inclined to the base of the specimen. Both the ultimate strength and the angle of fracture are to some extent influenced by the nature of the bedding material placed at each end of the test piece. Generally the harder the bedding material

the greater the force necessary to fracture the specimen and the more steeply inclined will be the planes of fracture; see Fig. 266. For example if milled lead be used as bedding material the planes of fracture will be nearly vertical, while if soft millboard be used the planes of fracture will approach the inclinations mentioned above. This effect is no doubt due to the restraint exercised on the ends of the specimen by the bedding material. It should therefore be clear that, in order to get comparative results in a series of tests, the same type of bedding material should be used throughout. Plaster of Paris, millboard and plywood are all suitable for this purpose. The ends of compression test pieces should be parallel to each other and at right angles to the axis. In the case of brick and stone it is not very difficult to cut the specimens by means of a simple arrangement similar to a stone saw. It may be noted here that built-up piers of brick or stone generally fail by splitting vertically. This fact appears to suggest that such piers would be strengthened by the insertion of lateral reinforcement in the joints.

Concrete. The inclined fracture is characteristic of compression failures in concrete, the shape of each portion of the fractured test piece being in the form of a pyramid in the case of cubical test pieces, and roughly conical in the case of cylindrical test pieces; see Fig. 266. Owing to the fact that the material is not homogeneous the planes of fracture will not usually be very well defined, unless a very small aggregate be used.

Experiment. An interesting test, showing the part played by the aggregate in resisting failure, may be carried out on 3 in. or 6 in. cubes, in which two specimens of concrete are tested, one with a very soft aggregate and the other with a very hard aggregate, the resulting fractures being compared.

The strength of concrete is of course affected by a great number of factors, among which are the nature and the proportions of the different materials used in its composition, the manner and conditions under which it is prepared and the age of the specimens. The subject is too lengthy and specialised for inclusion in this volume. The reader desiring further particulars should consult books dealing with concrete or with building materials generally.

Timber. See Chap. xxv.

Compression tests. If suitable facilities are available for testing compression specimens, it will be found to be both interesting and instructive to carry out a series of simple tests. The series should aim at making clear the manner in which failure takes place in the various materials. Attempts to get very accurate values or to produce stress-strain curves should, in the majority of cases, be

avoided at this stage. The elaboration of compression tests is in fact quite unnecessary for our present purpose.

If the specimens be not too large they may be tested on a compression testing machine similar to that shown in Fig. 267. This type of machine is oil-actuated, the load being put on by hand. Such machines may be capable of exerting pressures up to 50 or 100 tons.

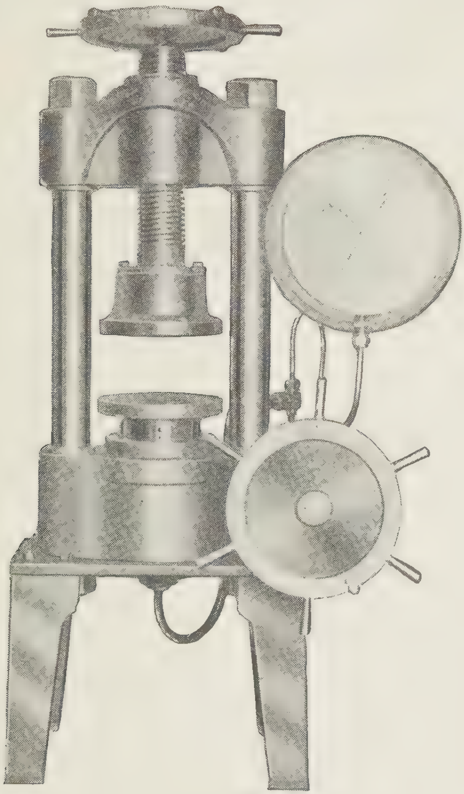


Fig. 267. Compression testing machine.

Average figures for the compressive strength of various structural materials are given in Tables VII and VIII, and in para. 225.

207. The strength of compression members—General terms. We have seen that, in the case of *very short* test pieces where the length (l) does not greatly exceed the width (d), failure in compression is by *crushing*.

If very *long and slender members* be tested they will be found to fail by *bending*, failure taking place at loads which would produce *direct stresses* much below the ultimate compressive strength of the material.

In the case of members of *medium slenderness* failure takes place by a combination of *bending and crushing*. In all but very short specimens we shall refer to this manner of failure as "failure by buckling". The load (P) at failure is known as the **Buckling load.** Then if A be the area of the section of the column, and we *assume the stress at failure to be uniformly distributed*, then

$$\frac{P}{A} = \text{buckling stress} = \frac{\text{buckling load}}{\text{area}}. \qquad \ldots\ldots\text{(i)}$$

Short columns. The three groups mentioned above cannot of course be sharply distinguished from each other. The definition of each group depends, as we shall see, in part upon the material used. It may, however, be taken that, unless special reference is made to the point, all columns or compression members—terms which may be taken to include walls and piers as well as struts—the length or height (l) of which is less than ten times the least lateral dimension (d), may be classed as **Short columns.** For **Short columns** with axial loads we can state at once that

$$\text{Safe load} = \text{area} \times \text{safe compressive stress}. \qquad \ldots\ldots\text{(ii)}$$

In the case of masonry and plain concrete structures the ratio between the length and the least lateral dimension, which may be expressed as l/d, may be at least 10 provided the rules governing masonry construction are observed.

Columns of medium and considerable slenderness. Great and apparently insuperable difficulties arise as soon as we attempt to obtain expressions which relate the strength of longer or more slender columns to their various dimensions. Many and varied theories have been advanced, which by their very multiplicity tend to confuse rather than to help the beginner. Moreover—and unfortunately as it would appear for our present purpose—no one solution can be said to have gained very general acceptance. The best that can be done is for the beginner to become thoroughly well acquainted with a reliable and simply stated theory, such as is given below. Following which he may, if needs be, extend his knowledge upon the basis here laid down, to include other more exact or more complex theories than can be dealt with in this volume.

208. Ratio of slenderness. A great simplification of the column problem was made by the introduction of the term "ratio of slenderness" or, more simply, the "slenderness" of a column. The strengths of all columns are found to bear a definite relation to this ratio, though the relation is one which cannot always be simply stated.

The ratio of slenderness includes the length (l) of the column and what is called the "radius of gyration" (g) of the section of the column. The slenderness of any given column is obtained by dividing the first term by the second, so that

$$\text{Slenderness} = \frac{\text{length}}{\text{radius of gyration}} = \frac{l}{g}. \qquad \text{......(iii)}$$

Radius of gyration. We have not had occasion to use this term before. It is obtained in the following manner. If I be the moment of inertia of any particular beam or column section, while A is the area of the section and g the radius of gyration, then

$$I = Ag^2. \qquad \text{......(iv)}$$

The term g is not capable of simple explanation but, just as in dealing with the first moment of an area, see paras. 14 and 15, we may think of the centroid as being the point at which the whole turning effect of the area is concentrated, so we may think of the resistance of the stressed area of a section as being concentrated at one point, that point being situated at some distance from the neutral axis, which distance we call "the radius of gyration" (g). The moment of inertia is in fact sometimes referred to as the "second moment of an area", in which case the analogy is complete. Thus we may say that, **the moment of inertia (I) of a section is equal to the area (A) multiplied by the square of the radius of gyration (g).**

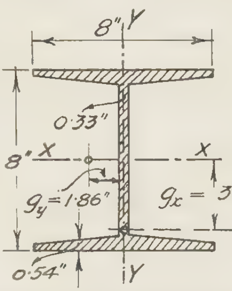


Fig. 268. Broad-flange beam—radius of gyration.

Let us consider an actual case. The section of an 8 in. × 8 in. broad-flange girder is shown in Fig. 268. The moment of inertia for this is $I_x = 124$ inch units^4 about the axis XX, while it is $I_y = 37{\cdot}7$ inch units^4 about the axis YY. Obviously these are the greatest and the least values of I for this section, hence, as we might expect, we shall get a maximum and a minimum value for g, the radius of gyration. From (iv) we have $I = Ag^2$, whence

$$g = \sqrt{\frac{I}{A}}. \qquad \text{......(v)}$$

In this case $A = 10{\cdot}9$ sq. ins. (This value and those for the moments of inertia are taken from a table supplied by the makers.)* Therefore

$$\text{the greatest value of } g = \sqrt{\frac{124}{10{\cdot}9}} = 3{\cdot}38 \text{ ins.} = g_x.$$

$$\text{Similarly the least value of } g = \sqrt{\frac{37{\cdot}7}{10{\cdot}9}} = 1{\cdot}86 \text{ ins.} = g_y.$$

These values are indicated on Fig. 268.

* Similar values for other sections are given in Appendix I.

From the above explanation the reader will see that the term g refers to a quantity, the magnitude of which varies with the manner in which the material is distributed about the neutral axis. Thus in the example just taken the value of g is greatest in that position in which the bulk of the material of the section is situated at the greatest possible distance from the neutral axis; see Fig. 268.

Hence it follows, since failure by buckling may be almost entirely due to bending, that

If a column is free to bend in any direction, then it is most likely to bend under load in that direction in which the radius of gyration (g) has its least value.

It is therefore the least value of g which we must use when calculating the ratio of slenderness; see example in para. 211 below.

Further, considered purely as columns under axial loads, those sections will obviously be most effective for which the value of g is the same in every direction; the only sections which completely satisfy this condition are the solid and hollow circular sections. For other sections the nearer they approach to this condition, the more efficiently will they resist buckling. Such practical questions as first cost, ease of fixing, accessibility for painting and inspection, usually cut out all such ideal sections except the solid circular section.

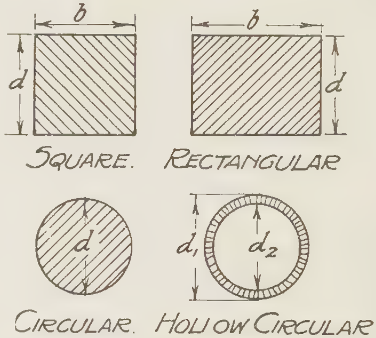


Fig. 269.

The following values of g for the geometrical sections will be found useful.

Square or rectangular section. Take d as the side of the square or the least dimension in the case of the rectangle, see Fig. 269,

$$I = \frac{bd^3}{12}, \quad A = bd.$$

Therefore $$g = \sqrt{\frac{I}{A}} = \sqrt{\frac{bd^3}{12bd}} = \sqrt{\frac{d^2}{12}} = 0{\cdot}289d. \qquad \ldots\ldots(\text{vi})$$

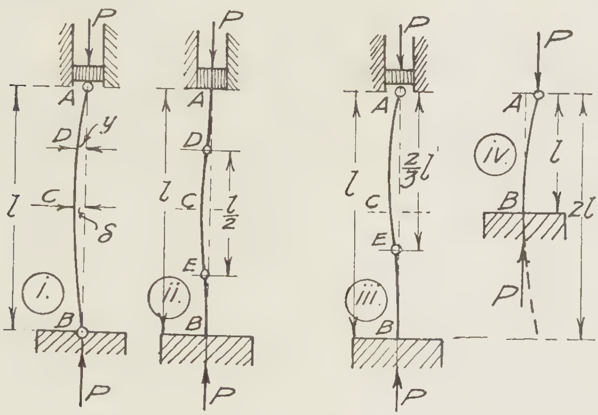
Circular section. In this case

$$g = \sqrt{\frac{I}{A}} = \sqrt{\frac{\pi d^4/64}{\pi d^2/4}} = \frac{d}{4}. \qquad \ldots\ldots(\text{vii})$$

Hollow circular section. By similar reasoning

$$g = \tfrac{1}{4}\sqrt{d_1^2 + d_2^2}. \qquad \text{......(viii)}$$

209. Strength of long columns—Euler's theory. For reasons which will presently appear the long or slender column is probably the most difficult type of column to design in practice. It is therefore rather surprising to find that we are in possession of a theory of the strength of this type of column which is probably both the oldest and the most widely accepted of the column



FIXING.	FREE BOTH ENDS	FIXED BOTH ENDS	ONE END FIXED. ONE END FREE.	ONE END FIXED. ONE END UNSUPPORTED.
EQUIVALENT LENGTH.	i. l	ii. $\frac{1}{2}l$.	iii. $\frac{2}{3}l$	iv. $2l$.
Note: The term VIRTUAL length is also used for EQUIVALENT.	i, ii, iii are "position fixed."			IV,-One end not "position fixed"

Fig. 270. The fixing of columns (ideal conditions).

theories. The theory referred to was propounded in the early part of the eighteenth century by a Swiss mathematician named Euler. Though the strict mathematical proof on which Euler's theory is based is too difficult to be included in this volume, the general statement is not difficult to understand and is, moreover, very informing as to the manner in which failure occurs in long columns.

Outline of proof. Let *ACB* in Fig. 270 (i) represent a very slender and perfectly straight column of uniform section, and let

it be assumed that, after a certain load P has been applied to the exact centre or axis of the column, it bends to the shape shown, so that it has a maximum deflection of δ at C and is in a state of equilibrium under these conditions. Then clearly the bending moment on the column at C will be $P\delta$.

Calling this bending moment B and *assuming* first that the bending moments on the column have the same value (B) at every point in the length l, then between A and B the column will bend to a circular arc and the deflection at C will be given by

$$\delta = \frac{Bl^2}{8EI}$$

(see para. 143), where E is the modulus of elasticity of the material of the column and I is the *least* moment of inertia of the section of the column.

Substituting $P\delta$ for B in this expression and transposing, then

$$P = \frac{8EI}{l^2}.$$

Now in this expression the terms E and I are constants while, if the column be very slender, the length l will also be practically constant, since δ will be relatively small. *Hence it follows that only one value of* P *can satisfy this equation.* Evidently if the load be less than P the column will tend to become less deflected if free to do so, while if the load exceed P then the deflection at C will go on increasing and the column will collapse. Experiments carried out under suitable conditions confirm this statement and it would therefore appear that P is what we have already called the **Buckling load.**

Reverting to our approximate expression for P, and considering Fig. 270 (i), it is clear that, once the column has bent, the value of the bending moment cannot be uniform throughout the length of the column since, if the deflection at D be y, then the bending moment at this section will be Py, and this is obviously less than $P\delta$. Our first assumption—that the column bent to a circular arc—is not, however, so very far from the truth since, by more exact mathematical reasoning, which allows for the variations in the bending moment, a similar expression is obtained but having π^2 as the multiplier in the place of 8, when

$$P = \textbf{buckling load} = \frac{\pi^2 EI}{l^2} = \frac{9{\cdot}87EI}{l^2}, \qquad \ldots\ldots\text{(ix)}$$

which is Euler's formula for free-ended columns.

The following example will explain how the formula is used in a simple case.

Example. A carefully prepared $\frac{3}{4}$ in. mild steel rod of circular section and 30 ins. long is to be tested as a free-ended column; find by means of Euler's formula the probable buckling load.

Here $\qquad l = 30$ ins. and $I = \dfrac{\pi d^4}{64} = \dfrac{\pi \times (0{\cdot}75)^4}{64}$.

Hence, if $\qquad E = 30{,}000{,}000$ lbs. per sq. in.,

$$P = \text{buckling load} = \frac{9{\cdot}87EI}{l^2} = \frac{9{\cdot}87 \times 30{,}000{,}000 \times 3{\cdot}14\,(0{\cdot}75)^4}{30^2 \times 64}$$

$$= 5100 \text{ lbs.}$$

It will have been noticed that the only term in expression (ix) which is related to the material used is E, the modulus of elasticity, and that the theory ignores the direct stress and rests solely on the idea of failure of the column by bending. It will not therefore be surprising to find that the theory is only supported by experimental results in those cases where the column is very long and slender, and where the direct stress must be small. For instance in the example just worked the value of g is given by $(d/4)$ or $\frac{3}{16}$ in., so that the ratio of slenderness is (l/g) or $(30 \times \frac{16}{3})$, that is 160, which indicates that this would be classed as a "long or slender column". We shall discuss at a later stage (see para. 213) just what are the limits within which Euler's formula can be practically applied.

210. Fixing the ends of columns. In deducing the Euler formula we assumed that the ends of the column were retained in such a manner as to exercise no restraint on the bending of the column, though the two forces P and P were assumed to remain in the same straight line. This manner of loading is represented diagrammatically in Fig. 270 (i), and the column is generally referred to as a "free-" or "round-ended column", or, less commonly, as "position-fixed", this term having reference to the fact that the ends A and B, though free to move vertically, are not free to move from the original straight line forming the axis of the column.

If now both ends of the column are "fixed"—or "direction- and position-fixed" as it should be called—as is shown in Fig. 270 (ii), the column will bend in a fashion similar to a beam fixed at both ends, there will thus be two points D and E at which the bending moment is zero, the axis of the column being straight at these points, where it is changing from one curve to another—points of contra-flexure; see para. 151.

It may be shown that, to apply Euler's formula to this and the following cases, we must use that portion of the column which is similar or equivalent to the round-ended column, this is called the **Equivalent Length.** In the case of the column fixed at both ends the "equivalent length" occurs between D and E, so that the

equivalent length is $(l/2)$ and this must replace "l" in finding the ratio of slenderness; see Fig. 270 (ii).

The two remaining cases should not require much further explanation, the equivalent length being indicated on the diagram. In case (iv) one end of the column is free to bend *in any direction*, the load remaining vertical, that is the end A is neither "position-fixed" nor "direction-fixed".

As will appear to the reader as we proceed, it is practically impossible to reproduce any of these conditions of end-fixing, even in the laboratory, with the perfection here indicated. In testing "free-ended" columns in a testing machine the load is usually transmitted to the specimen through hard steel balls or rollers, but even in such a case it can be shown that the result is affected by friction at the ends. If this occurs under the best of conditions, it will be easy for the reader practically acquainted with building to realise how difficult it must be to estimate the real fixity of columns used in buildings. Below we shall suggest certain modifications of the various formulae which may be adopted to meet this practical difficulty, see para. 215, but the problem is one requiring considerable knowledge and experience for its satisfactory solution. For the present the reader may adopt the above classification of columns unless some other method of procedure is indicated.

211. Safe or working loads. In order to arrive at a suitable safe load it is usual to divide the buckling load (P) by a factor of safety or, as we prefer to call it, a working factor (F), so that if

$$P = \text{buckling load,}$$

$$P/F = \text{safe or working load.}$$

Similarly, if A is the cross-sectional area of the column, then

$$P/AF = \text{safe or working stress.}$$

The value of F must vary, as we shall see, both with the material used and with the general conditions under which the column is employed. Suitable values will be indicated as we proceed. For mild steel columns a *minimum* value of 4 is usually adopted, though some higher value may have to be employed.*

Before working an example to show the application of the facts mentioned above, we may devise a modification of the Euler formula as given in expression (ix) which will be found to be of considerable practical use.

* In many column formulae a *sliding factor of safety* is used, so as to allow for the greater liability to original curvature in slender members. Generally the factor is $(4 + n \,.\, l/g)$, where n is arranged so that the factor equals 6 at the limit of slenderness where $l/g = 200$; hence $n = 0{\cdot}01$.

From (ix) we have

$$P = \frac{\pi^2 EI}{l^2}.$$

Substituting (Ag^2) for I in this expression and dividing P by A, we obtain what may be called the "average buckling stress," or more simply, the "buckling stress", then

$$\frac{P}{A} = \text{average buckling stress} = \frac{\pi^2 E\,(Ag^2)}{Al^2} = \frac{\pi^2 Eg^2}{l^2} = \frac{\pi^2 E}{\left(\frac{l}{g}\right)^2}. \qquad \ldots\ldots(\text{x})$$

Example. *The rolled section shown in Fig. 268 is to be used in the construction of a column 40 ft. long. Using a working factor of 4, find the safe working stress and the total safe load which the column will carry, if it can be assumed that it is rigidly fixed at one end and only position-fixed at the other.*

Using the values found in para. 208 we have A = area = 10·9 sq. ins., *least* radius of gyration (g) = 1·86 ins.

From Fig. 270 we have for the given conditions of fixing, that the equivalent length is $\frac{2}{3}$ of the full length or 320 ins. Hence

$$\text{Ratio of slenderness} = \frac{\text{equivalent length}}{g} = \frac{320}{1{\cdot}86} = 172.$$

From (x) we have

$$\frac{P}{A} = \text{average buckling stress} = \frac{\pi^2 E}{\left(\frac{l}{g}\right)^2} = \frac{9{\cdot}87 \times 30{,}000{,}000}{172 \times 172}$$

$$= 10{,}000 \text{ lbs. per sq. in.}$$

Then $\quad$ Safe stress $= P/AF = 10{,}000/4 = 2500$ lbs. per sq. in.

and $\quad$ Safe load = stress × area = 2500 × 10·9
= 27,250 lbs.

212. Strength of columns of medium length—Johnson's straight-line formula. As we have already indicated the failure of columns of medium length is partly by bending and partly by crushing, the latter being the more important in short columns. As might be expected, the attempt to express failure of so complicated a nature in an exact mathematical expression has led to the production of expressions of great length and complexity. As, however, the correctness of the theories so expressed are always checked by experimental results, it will serve our present purpose if we can find an expression of simple form which gives results reasonably close to those obtained by experiment. An expression satisfying these conditions in an admirable way is Johnson's Straight-Line Formula ("straight-line" because its graph gives a straight line, see next paragraph).

This formula was devised by T. H. Johnson of the U.S.A., after a lengthy series of tests and an investigation into the strength of columns of structural steel. The chief merits of this formula from our present point of view are (*a*) the simple form of the expression, (*b*) the fact that the constants used bear definite relations to the strength characteristics of the material, so that they may be based upon the result of direct experiments, (*c*)—following on (*b*) that similar expressions may be readily devised for other materials, and (*d*) that the results obtained (except for very low values of (l/g)) agree very well with experimental tests of columns.

The general form of the expression is as follows:

$$\frac{P}{A} = c_c - B\left(\frac{l}{g}\right), \qquad \ldots\ldots\text{(xi)}$$

where c_c and B are constants varying with the material. Suitable values of these constants for structural materials are given in Table XIII at the end of para. 216. and also in Table XV. We shall deal with each of the materials in turn and will, in addition, explain how the formula may be constructed in each case.

213. A complete T. H. Johnson-Euler graph for structural steel columns for all lengths (free-ends). The construction of this formula is best understood by studying the graph which covers the cases of both long and short columns. This graph is made up of a straight and a curved portion, the latter being obtained from Euler's formula; see Fig. 271.

Briefly the theory underlying this formula is: (a) that the strength of very short columns corresponds to the crushing strength c_c of the material, (b) that the buckling strength of long columns is satisfactorily expressed by Euler's formula, and (c) that between these two stages the buckling strength falls off uniformly from one value to the other.

In Fig. 271 is plotted a curved line CDG which shows how P/A, the buckling load, varies according to Euler's formula with the slenderness ratios (l/g), which are plotted along the horizontal axis. It follows from the above statement that the graph of the Johnson Straight-Line Formula is a straight line AD which starts at A, where the slenderness ratio is zero, and the stress equal to the crushing stress c_c of the material, and is drawn tangential to the Euler curve at D. The stress at A which Johnson recommends for structural steel is 52,250 lbs., which is somewhat lower than the actual crushing strength of such steel. Rather better results are in fact obtained in the case of ductile materials and with timber, if this stress is between 80 % to 90 % of the actual crushing stress. *The stress at* A *gives the value of the constant* c_c in expression (xi).

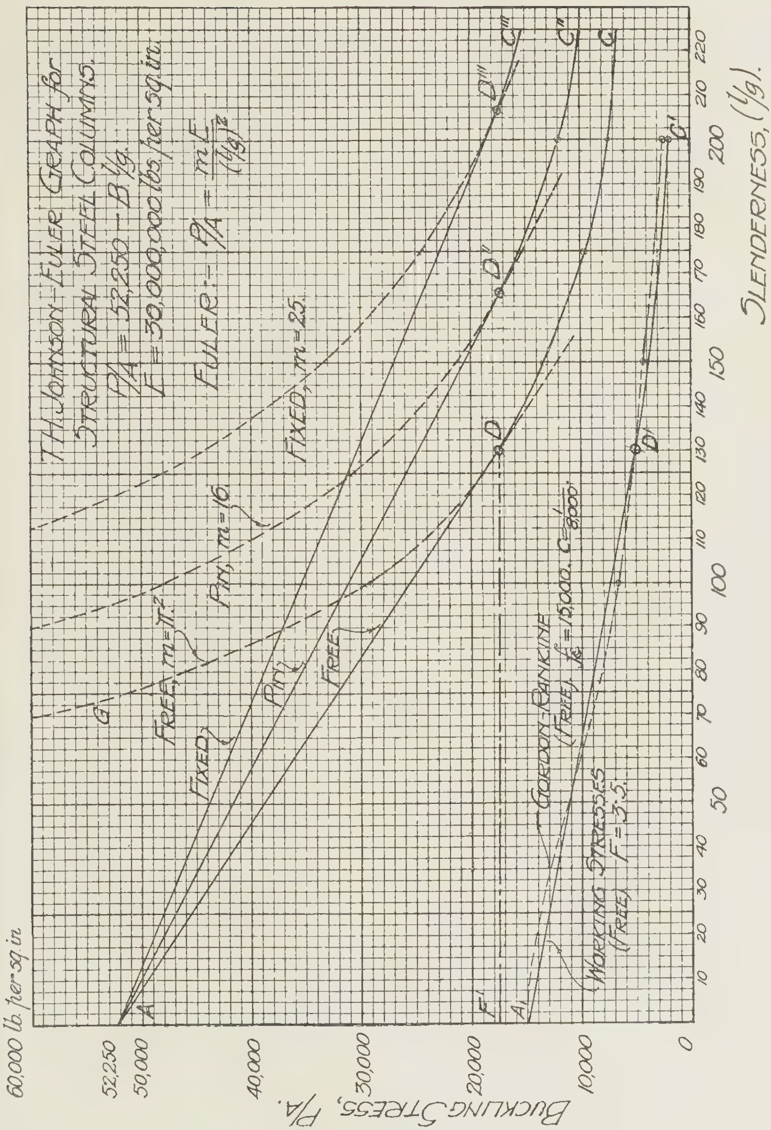


Fig. 271. T. H. Johnson-Euler graph for structural steel columns.

The complete graph for *all* values of (l/g) is then given by the line ADC. The straight portion AD will be sufficiently accurate if it is merely drawn to touch the curve CDG, after the latter has been obtained. *The value of the constant* B is then readily obtained by the usual graphical method of finding the second constant in a linear equation. In this case

$$B = AF'/F'D \qquad \text{(see Fig. 271)}$$
$$= 35{,}000/130 = 270,$$

which is the value given in Table XIII at the end of para. 216. The limiting value of l/g indicated in the Table gives the position of the tangent point D in each case, and so indicates the "limit" beyond which the straight-line graph should not be used, Euler's values being used beyond this point. In this case the limit of l/g is 130.*

The two formulae for structural steel are then

$$(a),\ \frac{P}{A} = 52{,}250 - 270\left(\frac{l}{g}\right),\ \text{up to a limit of } \frac{l}{g} = 130,$$

when $$(b),\ \frac{P}{A} = \frac{\pi^2 E}{\left(\frac{l}{g}\right)^2}. \qquad \text{......(xii)}$$

* The values of the constants may be deduced mathematically as follows. From $P/A = c_c - B\,(l/g)$ and Fig. 271, it will be seen that B must have such a value as to make the straight-line graph tangential to the Euler curve. The equation for the latter should be put in the form $P/A = mE/(l/g)^2$, where m is a constant of which the value is π^2 in the ordinary Euler formula.

At the tangent point D both ordinates have the same value, hence

$$\frac{mE}{\left(\frac{l}{g}\right)^2} = c_c - B\left(\frac{l}{g}\right),$$

whence $$mE = c_c\left(\frac{l}{g}\right)^2 - B\left(\frac{l}{g}\right)^3. \qquad \text{......(a)}$$

Also differentiating $P/A = mE/(l/g)^2$ with respect to (l/g), we have

$$\frac{d\,(P/A)}{d\,(l/g)} = -\frac{2mE}{(l/g)^3},$$

which equals $-B$ at the tangent point, and from which we have

$$2mE = B\,(l/g)^3. \qquad \text{......(b)}$$

Adding (a) and (b) we have $3mE = c_c\,(l/g)^2$, hence the tangent point occurs where

$$l/g = \sqrt{\frac{3mE}{c_c}}. \qquad \text{......(c)}$$

Also from (b) we have

$$B = \frac{2mE}{\left(\frac{l}{g}\right)^3}. \qquad \text{......(d)}$$

As is explained later the multiplier m must be given other values than π^2 when the ends are not free.

214. Working stresses from the Johnson-Euler curve. The graph of the formula which we have just plotted indicates the buckling loads (P/A) for a free-ended column. To find the safe loads on such a column we must reduce these values by means of a factor of safety or working factor (F) as already explained. This factor is obtained as follows. The safe compressive stress which can be put on a very short column is usually known or can be ascertained by experiment, then from this we have

$$F = \text{working factor} = \frac{\text{crushing strength } (c_c)}{\text{safe compressive stress}} \quad \text{......(xiii)}$$

For mild steel columns a safe compressive stress of 15,000 lbs. per sq. in. may be allowed. We thus have

$$F = \frac{52{,}250}{15{,}000} = 3{\cdot}5.$$

Using this value for F a second curve $A'D'G'$ has been drawn in Fig. 271 which gives the working stresses for free-ended mild steel columns.

215. Allowance for end-fixing in Johnson-Euler curves. Instead of attempting to follow the "ideal" plan laid down in para. 210 with regard to the effect of end-fixing, Johnson proposed to modify the curved part of the graph by altering the constant in the Euler formula. If the latter is expressed as

$$\frac{P}{A} = \frac{mE}{\left(\frac{l}{g}\right)^2}, \quad \text{......(xiv)}$$

where m is a constant, then Johnson proposed that m should have the following values:

(a) Ideally free ends $m = \pi^2$,
(b) Pin joints at ends $m = 16$,
(c) Fixed ends $m = 25$.

These constants were deduced after considerable experience with practical columns and may be used with confidence. In effect they show that a "practical" pin joint is not the same as an ideal "free-end", owing to friction at the pin under load. Similarly it may be shown that "practical" fixing is less effective than "ideal" fixing. It should be noted that with this method of calculation, it is not necessary to find the "equivalent" length of the column.

No matter how a column is fixed the crushing stress, if the column be *very* short, will not be affected; hence Johnson commences the straight-line part of these additional curves at the same point A

as before. The straight lines AD'' and AD''' are drawn tangentially to the corresponding Euler curves as before, the complete curves being given by $AD''C''$ and $AD'''C'''$ respectively in Fig. 271. The corresponding values for use in the formula are given in Table XIII at the end of para. 216.

The two further expressions thus obtained for **Mild Steel Columns** may be set out here:

Pinned ends.

$$(a)\quad \frac{P}{A} = 52{,}250 - 210\left(\frac{l}{g}\right), \text{ up to a limit of } \frac{l}{g} = 166,$$

$$\text{when}\quad (b)\quad \frac{P}{A} = \frac{16E}{\left(\frac{l}{g}\right)^2}. \qquad \text{......(xv)}$$

Fixed ends.

$$(a)\quad \frac{P}{A} = 52{,}250 - 170\left(\frac{l}{g}\right), \text{ up to a limit of } \frac{l}{g} = 207,$$

$$\text{when}\quad (b)\quad \frac{P}{A} = \frac{25E}{\left(\frac{l}{g}\right)^2}. \qquad \text{......(xvi)}$$

Flat ends. In some cases beams merely *rest* upon the flat ends of columns, or columns *rest* upon a foundation, or are only lightly secured thereto. In these cases the columns may usually be treated as fixed columns *up to a certain point*, beyond which it is safer to treat them as pin- or free-ended columns. This limit for "flat-ended" columns cannot be fixed easily, it should be very low for brittle materials and in any case the limit should err on the side of safety. Suitable limits have been indicated in Table XIII.

It may be noted here that in practice the maximum allowable slenderness of columns is usually fixed by regulation. For steel columns this is generally from 100 to 150 for free-ended columns, and from 150 to 200 for fixed columns.

The following Examples will further explain how the values obtained from Fig. 271 are to be utilised in the solution of problems.

Example 1. *A solid circular steel column, 4 ins. in diameter, has solid flanged ends and is used to support a beam at the centre. The beam carries the front of a building. The length of the column is 15 ft. and it may be assumed from the way in which the ends of the column are secured that it is a fixed column. Find the safe load which it will carry.*

In this case the length $l = 180$ ins.

The value of g = diameter/4 = 1 in. Therefore the slenderness ratio is 180.

From the appropriate graph in Fig. 271 we have, where l/g is 180,

$$P/A = 22{,}000 \text{ lbs. per sq. in.}$$

Therefore, if the value of F is 3·5,

$$\text{Safe stress} = 22{,}000/3{\cdot}5,$$

and
$$\text{Safe load} = \text{stress} \times \text{area} = \frac{22{,}000}{3{\cdot}5} \times \frac{\pi\,(4)^2}{4}$$
$$= 79{,}000 \text{ lbs.} = 35 \text{ tons.}$$

Example 2. *In order to construct a scaffolding it is proposed to utilise steel tubing of* $1\frac{3}{4}$ *ins. outside diameter, the metal being* 5 S.W.G. (0·212 *in.*); *see Fig.* 272. *If the maximum unsupported length of uprights is to be* 7 *ft. and the tubes are clipped together at the crossings by special clips with set screws, find the safe maximum load which may be put upon an upright, if the safe maximum stress allowed on a short length is* 10,000 *lbs. per sq. in.*

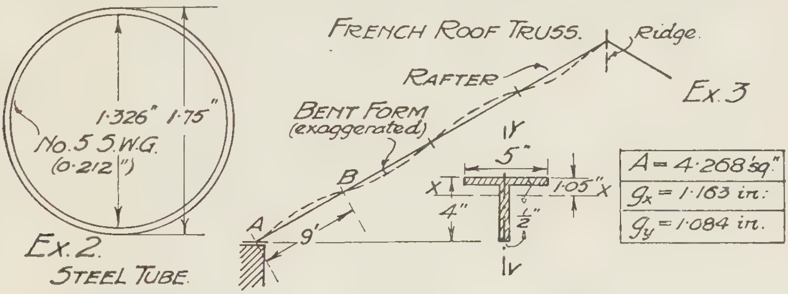


Fig. 272.

Since it is clear that, though some of the tubes may be continuous through several vertical bays of the scaffolding, others may be used in lengths only slightly over 7 ft., then it will be advisable to assume "free" ends in every case, especially in view of the non-rigid nature of the connections.

In this case

$$F = \text{working factor} = \frac{52{,}250}{10{,}000} = 5{\cdot}22.$$

$$\text{Area of tube section} = \frac{\pi}{4}\,(1{\cdot}75^2 - 1{\cdot}326^2) = 1{\cdot}01 \text{ sq. ins.}$$

Also $\quad$ Least value of $g = \frac{1}{4}\sqrt{1{\cdot}75^2 + 1{\cdot}326^2}$ $\quad$ (see (viii), para. 208)
$$= 0{\cdot}55 \text{ in.}$$

Then $\quad$ Slenderness $= l/g = 84/0{\cdot}55 = 153.$

Then, from Fig. 271, $P/A = 12{,}750$, whence

$$\text{Safe load} = 12{,}750 \times 1{\cdot}01 \div 5{\cdot}22 = 2470 \text{ lbs. (approx.).}$$

Example 3. *In the French roof truss shown in Fig.* 159 *the maximum compressive force coming on the lower portion* AB *of the principal rafter, after allowing for dead and wind loads, is* 12 *tons. Ascertain suitable dimensions for this member. The length of the member is* 9 *ft.* See Fig. 272.

In dealing with the forces acting in this truss we assumed that all the joints were pin-jointed. In practice, however, the rafter is almost invariably continuous from the ridge to the eaves. Though the con-

nections at the eaves and at the ridge may be fairly substantial the intermediate connections are much less rigid so that, in all probability, the rafter bends with a fair measure of freedom between these connections; see Fig. 272. Thus, in spite of the fact that this member is continuous over several spans, the choice seems to lie between "free" and "pinned" ends, rather than between "pinned" and "fixed" ends. In view of the fact that part of the load—the wind load—is a "live" load it will probably be better to assume "free" ends, in effect to allow for the presence of the "live" load by making a reduction in the working stress.

Usually several trials will have to be made before a suitable section will be found, the procedure in each case being the same as that set out below. A T-section 5 ins. by 4 ins. by $\frac{1}{2}$ in. may be tried; see Fig. 272. The various dimensions of the section have been taken from the Tables given in Appendix I.

The area of the section $= A = 4{\cdot}268$ sq. ins.

The least radius of gyration is seen to be about the $Y\!-\!Y$ axis, the value being 1·084 ins.

Then $$\text{Slenderness} = \frac{108}{1{\cdot}084} = 100.$$

From Fig. 271, $P/A = 25{,}500$ lbs.

Hence, if $F = 3{\cdot}5$, $$\text{Safe load} = \frac{25{,}500 \times 4{\cdot}268}{3{\cdot}5 \times 2240} \text{ tons}$$
$$= 14 \text{ tons.}$$

This appears to give a satisfactory margin of strength (but see Example 1, para. 218).

216. Cast iron columns. Owing to their liability to fracture without warning under excessive or accidental loads, very slender cast iron columns are very rarely used. Nor are they used frequently in positions where the fixing may be described as either "free" or "pinned". For the remaining and more frequent case of a "fixed-" or "flat-" ended cast iron column the T. H. Johnson-Euler formula, taking the values given in Table XIII, becomes

$$P/A = 80{,}000 - 438\,(l/g). \qquad \text{......(xvii)}$$

A low safe working stress of 10,000 lbs. per sq. in. may be used, to allow for the difficulty of ensuring that the castings are perfectly sound; this makes F, the working factor, take a value of 8.

A value of $l/g = 122$ is given in the Table as the limit of the straight portion of the graph. Cast iron columns should not usually be used for so high a degree of slenderness, which it is more satisfactory to limit to 80. It will thus be seen that no part of the Euler curve need be used and the above formula is sufficient. E may be given a value of 15,000,000 lbs. per sq. in., if the graph is to be constructed from the Euler curve.

Table XIII

T. H. Johnson-Euler Column Formulae. Table of Constants

I. *Euler portion of graph.* $P/A = \dfrac{mE}{(l/g)^2}$.

Values of m: Free (ideal), π^2 (= 10 approx.); Pin, 16; Flat or fixed, 25.

II. *Straight-line portion.* $P/A = c_c - B\,(l/g)$.

Material and fixing	c_c lbs. per sq. in.	B	Upper limit (l/g)	Maximum working stress	Working factor F
Structural steel. (E = 30,000,000 lbs. per sq. in.)					
Free	52,250	270	130	15,000 (or $6\frac{1}{2}$ tons)	3·5
Pin	52,250	210	166	15,000	3·5
Fixed or Flat (up to l/g = 100)	52,250	170	207	15,000	3·5
Cast iron. (E = 15,000,000 lbs. per sq. in.)					
Free	80,000	693	77	10,000	8
Pin	80,000	537	99	10,000	8
Flat (to l/g = 80)	80,000	438	122	10,000	8

Timber Columns. See para. 227.

217. Other column theories. Having now considered with some degree of thoroughness a simple but fairly complete theory of column strength, the reader should extend his survey as opportunities offer to include other theories not dealt with here. In addition he may with advantage study some of the well-known building regulations affecting column design, as well as a few good examples of columns actually erected under those regulations. In this way he will acquire some power to discriminate between those refinements which are not out of place in the laboratory, and the practical necessities of economic construction which the designer and constructor have to keep in mind. Of the many theories which might have been touched upon we will refer briefly only to one.

Rankine-Gordon column formula. This formula, which has had a long and useful history in this and other countries, was devised by Rankine as a modification of an earlier formula constructed by Gordon. In one respect it is similar to the Johnson-Euler graphs which we have been discussing, in that it is based upon the strength of very short columns at zero slenderness, and upon Euler values at the other end of the scale. While its form may be deduced mathematically, it contains a constant (a) the value of which is based upon experiments not easily carried out, and this may in part

explain why the values published by different authorities are so varied. Put in a form to give safe working stresses the formula is as follows:

Rankine-Gordon column formula. Free ends

$$p = \text{safe working stress} = \frac{\text{maximum safe stress}}{1 + a\left(\dfrac{l}{g}\right)^2} \quad \ldots\text{(xviii)}$$

The following values of the constants are frequently adopted:

Table XIV

Rankine-Gordon Column Formula. Table of Constants

Material	Maximum safe stress	Constant (a)
Mild steel	15,000 lbs. (or $6\frac{1}{2}$ tons) per sq. in.	1/8000
Cast iron	10,000 lbs. per sq. in.	1/3000

(Note. The above values are for "free-ended" columns; for other columns the "equivalent length" must be used; see para. 181. A graph (dotted) of the above formula for mild steel is given in Fig. 271.

Problems XXIII

1. A convenient form of extensometer for use with large test pieces (tension or compression) is known as Marten's, which makes use of the principle of the "optical lever" described in para. 144, in connection with the measurement of the slope of a beam. The extensometer in one of its forms is shown in Fig. A in use on a compression test of a concrete column. The extensometer consists of a bar of steel, having at the end A a knife-edge, while at B an accurate V-groove is formed in which rests a hard steel diamond-shaped fulcrum. The length AB is the gauge length, in this case 12 ins. The exact width bb of the fulcrum is measured (x), and in this case is 0·5 in. A small mirror is attached to the fulcrum and rotates through an angle θ as the fulcrum is tilted by the shortening (or lengthening) of the specimen; see Fig. A. The extensometer is held in position by a spring clip or by means of rubber bands. As explained in para. 144, a distant scale is viewed by reflection in the mirror by means of a telescope having a cross hair, the reflected ray rotating through an angle of 2θ. If the distance to the scale from B is 10 ft. and the ray moves a distance X (= 1·7 ins.) over the scale, calculate the total shortening in the length l and the stress in the specimen, if E is known to be 2,000,000 lbs. per sq. in.

2. Calculate the ratio of slenderness of a column 20 ft. long which is made from a 9 in. by 7 in. by 50 lbs. rolled steel section (see Appendix I for other particulars), (a) as a free-ended column, and (b) as a column fixed at one end but only position-fixed at the other.

3. Calculate the buckling load on a test column, using Euler's formula, the test piece being loaded through steel balls at each end. The column is 28 ins. long, 0·5 in. in diameter and of structural steel.

4. Using the values given in Table XIII, find the safe load on a strut 9 ft. long, made from a T-section 4 ins. by 4 ins. by $\frac{1}{2}$ in., (*a*) if the ends be considered as pinned ends, and (*b*) if the ends are assumed to be fixed ends.

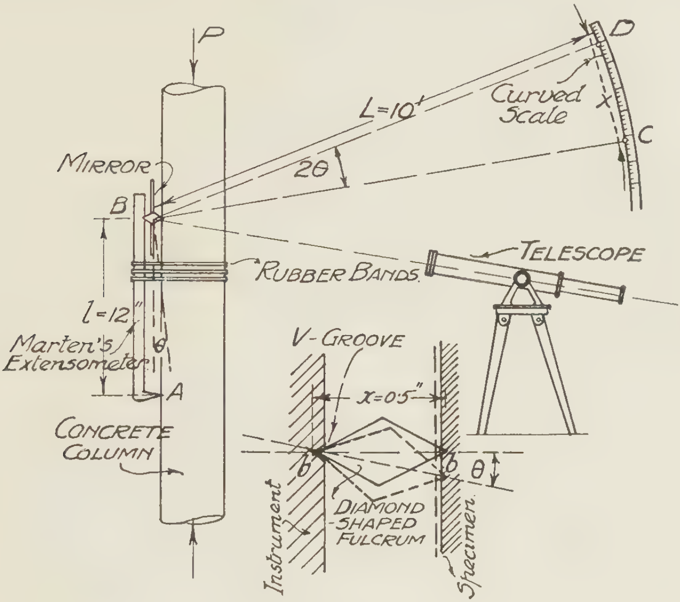


Fig. *A*.

5. Find the safe load on a cast iron column of hollow-circular section, 8 ins. outside diameter, 6 ins. inside diameter and 11 ft. long with flat ends, (*a*) using the Johnson-Euler formula and Table XIII, and (*b*) using the Rankine-Gordon formula and Table XIV.

CHAPTER XXIV

ECCENTRIC LOADS ON COLUMNS. REINFORCED CONCRETE COLUMNS. COLUMN FOUNDATIONS

218. Effect of eccentric loads on columns. Up to the present, we have restricted our treatment of the strength of columns to those cases of *axial loading* in which the stresses induced in the column could be taken to be uniformly distributed over the section. As we saw in the case of masonry construction (see Chap. XXI), this uniform distribution of stress ceases to hold as soon as the load becomes non-axial or eccentric. In such cases the maximum stresses induced may greatly exceed those induced by an axial load of the same magnitude.

In practice, eccentric loads on columns may arise in a variety of ways (see Examples given below), and since, as we shall see, the effect of even a small amount of eccentricity is very marked, it is most important that we should be able to calculate the magnitude of the stresses so induced, and also know what are the safe limits within which they may lie. The problems divide themselves into two main groups; those dealing with short columns, and those dealing with slender columns.

Short columns with eccentric loads. When dealing with eccentric loads on masonry we found the maximum and minimum stresses by means of the expressions (iv) and (v) in para. 190. These may be used also in the case of short columns and may be re-written as follows:

$$\text{Maximum compression} = \frac{P}{A} + \frac{Pe}{Z}, \qquad \ldots\ldots\text{(i)}$$

$$\text{Minimum compression} = \frac{P}{A} - \frac{Pe}{Z}, \qquad \ldots\ldots\text{(ii)}$$

where P is the load applied to the column, A is the area of the section of the column, while Z is the section modulus of the column taken about a line through the centroid at right angles to the line along which the eccentricity e is measured.

In cases where *the second expression gives a negative result it will indicate that the stress induced on the edge farthest from the load is one of tension.* Since the material may therefore have to resist tensile as well as compressive stresses, our first condition becomes:

I. Both the maximum compressive and tensile stresses must be kept within the safe working stresses of the material.

Slender columns with eccentric loads. In the case of long or slender columns a more exact expression may be deduced, by considering the effect of eccentricity on the expression given for the Euler load on the column. Alternately a type of column formula may be used, such as Moncrieff's, in which the effect of the eccentricity may be readily allowed for. In our case, however, it will be sufficient in the case of long columns to adopt the expressions given above for short columns, with the added condition that

II. The maximum compressive stress due to eccentric loading must not exceed the safe working stress obtained from the buckling load for an axially loaded column of the same length and fixing.

The following examples will explain the procedure to be followed.

Example I. In calculating the strength of the principal rafter in Example 3, para. 215, we assumed that the load was axially applied. From Fig. 272 we see that the axis X–X crosses the axis Y–Y at a distance of only 1·05 ins. from the upper surface of the T-section. This distance is obviously too small to allow of the line of rivets forming the connection to the rafter being placed on the axis. Let us assume that the rivets are placed on a line 2 ins. from the upper flange of the T, then the value of e, the eccentricity, will be equal to $(2 - 1{\cdot}05)$ or 0·95 in. In this case P the load on rafter = 12 tons, and A the area of section = 4·268 sq. ins. Also, from the Tables given in Appendix I, the value of Z, the modulus of section, about X–X is 1·96 inch units^3. Then

$$\text{Maximum compression} = \frac{P}{A} + \frac{Pe}{Z}$$

$$= \frac{12 \times 2240}{4{\cdot}268} + \frac{12 \times 2240 \times 0{\cdot}95}{1{\cdot}96}$$

$$= 19{,}300 \text{ lbs. per sq. in.}$$

From the previous Example, para. 215, we have

$$\text{Safe average stress} = \frac{25{,}500}{3{\cdot}5}$$

$$= 7300 \text{ lbs. per sq. in. (approx.).}$$

The maximum stress due to the eccentric load is thus not only greatly in excess of that due to the same load when applied axially, but is also in excess of the safe maximum stress (15,000 lbs. per sq. in.) which can be put upon this material. In order to allow the rivets to be placed along the axis of the rafter, or as near to it as possible, two L-sections placed back to back are sometimes used in this case. A trial might be made with two 5 ins. × $3\frac{1}{2}$ ins. × $\frac{3}{8}$ in. angles, the rivets being placed 3 ins. from the lower edge, giving an eccentricity of less than $\frac{1}{4}$ in.; see Fig. 273 (i). (The calculations are left as an exercise for the reader; see Problems XXIV, 3.)

Example 2. *Let the stanchion dealt with in the Example given in para.* 208, *and shown in Fig.* 268, *be used to carry loads from the girders* A, B *and* C *as shown in plan in Fig.* 273 (ii). *The magnitudes of these loads and also the positions of the points at which they may be taken to act (obtained by a consideration of the way in which they are supported, in*

this case by simple brackets, see para. 223) are indicated in Fig. 273 (ii). *Ascertain whether the maximum compressive stress induced by these eccentric loads exceeds the safe stress which can be allowed on such a stanchion for axial loading, assuming that it is* 25 *ft. long and fixed at both ends.*

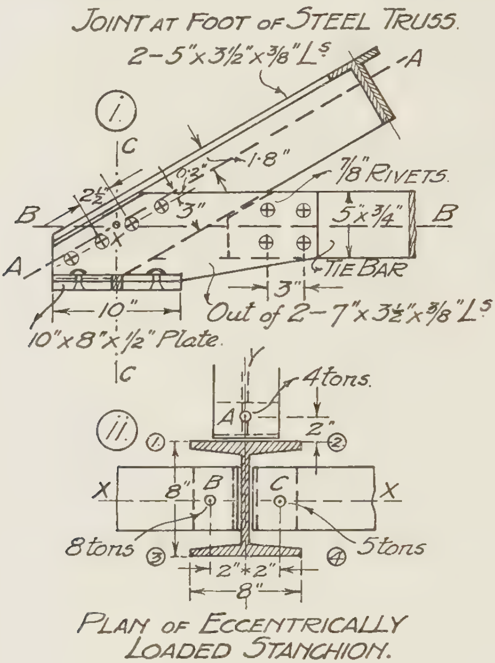


Fig. 273.

We will first find the safe axial load. From the Example given in para. 208 we have the following values:

$$\text{Radius of gyration} = g = 1{\cdot}86 \text{ ins.}$$

$$\text{Area} = A = 10{\cdot}9 \text{ sq. ins.}$$

$$\text{Section modulus about } X\text{–}X = Z_{xx} = \frac{I_{xx}}{4} = \frac{124}{4} = 31 \text{ inch units.}$$

Similarly $$Z_{yy} = \frac{37{\cdot}7}{4} = 9{\cdot}4 \text{ inch units.}$$

Then $$\frac{l}{g} = \frac{25 \times 12}{1{\cdot}86} = 160.$$

From Fig. 271 or from expression (xvi), Chap. XXIII, we have

$$P/A = 52{,}250 - 170 \times 160 = 25{,}050 \text{ lbs.}$$

Hence, if $F = 3{\cdot}5$, we have

$$\text{Safe stress} = \frac{25{,}050}{3{\cdot}5} = 7157 \text{ lbs. per sq. in.}$$

We find the stresses due to each eccentric load as follows: Taking load A of 4 tons first, the value of e is $(4 + 2)$ or 6 ins. The value of Z_{xx} as we have seen is 31. Then

$$\text{Maximum compressive stress} = \frac{4 \times 2240}{10 \cdot 9} + \frac{4 \times 2240 \times 6}{31}$$

$$= 820 + 1730 = 2550 \text{ lbs. per sq. in.}$$

and this stress acts across the flange (1)–(2).

Similarly the minimum stress $= 820 - 1730 = -910$, which indicates that the stress acting over the flange (3)–(4) is a tensile stress of this magnitude.

The loads B and C act on the same axis X–X and may therefore be combined in one load of $(8 + 5)$ or 13 tons. Let this load act at a distance x from B, then we may find x by taking moments about B, when we have $x \times 13 = 5 \times 4$, or $x = 1 \cdot 54$ ins.

The eccentricity of the load of 13 tons is thus $(2 - 1 \cdot 54)$ or $0 \cdot 46$ in. from the Y–Y axis.

The value of Z_{yy} being 9·4, we have

Maximum stress due to combined load (B and C)

$$= \frac{13 \times 2240}{10 \cdot 9} + \frac{13 \times 2240 \times 0 \cdot 46}{9 \cdot 4} = 2675 + 1430$$

$$= 4105 \text{ lbs. per sq. in. (approx.),}$$

and this stress acts at the corners (1) and (3).

The minimum stress due to this load $= 2675 - 1430$

$$= 1245 \text{ lbs. per sq. in. (approx.),}$$

and this acts at corners (2) and (4).

The maximum stress, by addition, is thus seen to act at the corner (1) where it $= 2550 + 4105 = 6655$ lbs. per sq. in., which is less than the average stress for an axial load (7157).

The minimum stress acts at corner (4) and equals $(1245 - 910)$ or 335 lbs. per sq. in. compression.

219. Miscellaneous notes on columns, etc.

Note on joints in trusses. It will be recalled that in Chaps. v and xi, where we dealt with framed structures, we made the assumption that the joints were pinned joints. Except in very large and important structures such joints are rarely used in this country. In order to ensure, however, that the actual conditions at the joint do not depart too markedly from the assumed conditions, the axes of the various members meeting at a joint are made to intersect at a point. Thus in Fig. 273 (i) if A–A be the axis of the principal rafter, then B–B the axis of the tie bar and C–C the axis of the bearing plate of the truss are all made to intersect at the point X. (The size and spacing of the rivets and the dimensions of the cover plates for such a joint are determined by the rules laid down in Chap. xviii for riveted connections.)

Accidental eccentricity. If a column be bent, or if, being straight, the quality of the material of which it consists varies from place to place, then, even if the load be applied axially at the ends, the

effect will be the same as for an eccentric load. This we may call "accidental eccentricity". It is more frequent than may appear at first sight and, in order to allow for it in practice, some of the modern column formulae– of which Moncrieff's is a good example*— deliberately assume a definite amount of eccentricity in calculating the loads which a column will carry. The condition may be said to be covered in the formulae we have used above by adopting a fairly high reduction factor or factor of safety (F).

Combined bending and direct thrust. If, in the rafter dealt with in Example 3, para. 215, a load were applied transversely to the rafter at some point between the joints at the ends, then, in addition to the compressive stress set up by the direct load, there would also be tensile and compressive stresses set up as in a beam. To find the maximum stresses in such a case we add together, as in the case of an eccentrically loaded column, the stresses due to the direct load and those due to bending. The expressions (i) and (ii) given in this chapter may be used but should be re-written thus:

Maximum compressive stress

$$= \frac{\textbf{total direct thrust}}{\textbf{area}} + \frac{\textbf{bending moment}}{\textbf{modulus of section}}. \qquad \ldots\ldots\text{(iii)}$$

The minimum stress (or tensile stress if the result be negative) would be similarly obtained by changing the sign in the expression.

(The direct stress may of course be a tensile stress, in which case "direct tension" is substituted for "direct thrust" in the above expression; a negative result in this case would indicate compression.)

Cases where expression (iii) may be applied occur when purlins are carried on principal rafters at points between the joints, in columns which have to resist a side thrust, and in reinforced concrete retaining walls or chimneys. The effect is similar to that of an eccentric load on a column or a wall, and only differs from the latter in that the member is able to resist tension.

Reinforced Concrete Columns

220. Rectangular reinforced concrete columns—Short columns. It would be possible to use very short columns in plain concrete to carry *axial* loads, but it would only be safe to do so if the circumstances were such as to ensure that there could be no considerable departure from the condition of axial loading, *otherwise* the concrete might be subjected to tensile stresses and fail in consequence. In all ordinary cases such columns are now reinforced. This reinforcement may be conveniently considered in two parts: (a) the vertical or longitudinal reinforcement, and (b) the lateral reinforcement, "binding" or "hooping".

* See Husband and Harby's *Structural Engineering.*

(a) **Vertical reinforcement.** Typical sections of square and rectangular columns in reinforced concrete are shown in Fig. 274. As a rule the amount of vertical reinforcement should not be less than 1 %, as reckoned on the sectional area of the column, while the bars should not be less than $\frac{1}{2}$ in. nor more than 2 ins. in diameter. The vertical reinforcement is covered on the outside by $1\frac{1}{2}$ ins. to 2 ins. of concrete. Some authorities look upon this "cover" as a necessary protection in the case of fire and do not allow it

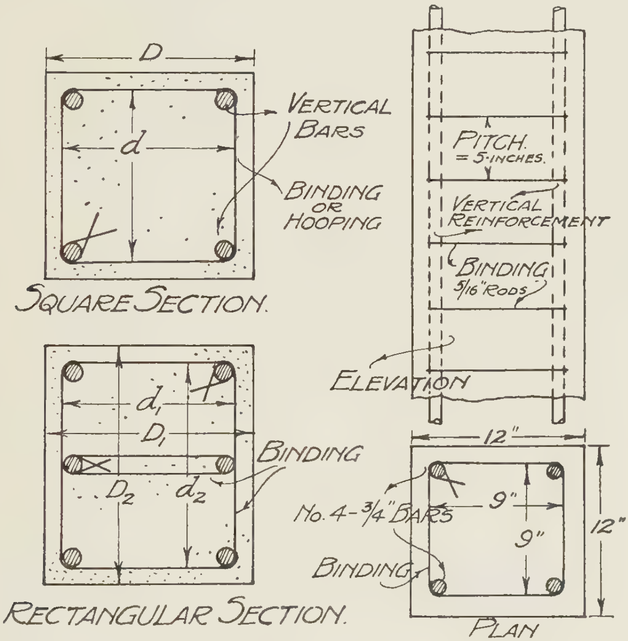


Fig. 274. A reinforced concrete column.

to be included in calculating the strength of the column. Thus if, in Fig. 274, D_1 and D_2 be the outside dimensions, or "diameters" as they are usually termed, then d_1 and d_2, measured to the outside of the vertical reinforcement, are known as the "effective diameters". The "effective area" (A) is then given by ($d_1 \times d_2$).

Safe load on a short column. Concrete columns are designed on the assumption that the concrete and the vertical bars are strained or shortened together under load, so that, if m be the modular ratio, see Chap. XIX, and c be the stress in the concrete surrounding

the bar, then the stress in the bar will be equal to mc. For normal "1 : 2 : 4" concrete c is usually 600 lbs. per sq. in. and m is 15, hence the maximum stress in the steel will be 9000 lbs. per sq. in.

It follows from this that the "equivalent area" (see paragraphs 169 and 179 (A)) of a column section will be

$$A_e = \text{equivalent area} = A + (m - 1)\, A_v, \qquad \text{......(iv)}$$

where A_v is the sectional area of the vertical reinforcement.

Then, if c be the safe compressive stress on the concrete, the

$$\text{Total safe load} = cA_e = c\,\{A + (m - 1)\, A_v\}. \qquad \text{......(v)}$$

Example. In the dimensioned section shown in Fig. 274 the effective area of the column $= A = 9 \times 9 = 81$ sq. ins.

Total area of vertical reinforcement $= A_v = 4 \times 0{\cdot}442$
$= 1{\cdot}768$ sq. ins.

Then Equivalent area $= A_e = 81 + (14 \times 1{\cdot}768) = 106$ sq. ins.
Total safe load $= c \times A_e = 600 \times 106 = 63{,}600$ lbs.

(b) **Effect of lateral binding.** Since the vertical reinforcement, which is in compression, is comparatively close to the outer surface of the column it is clear that the vertical reinforcement must be tied into the body of the column, otherwise it may buckle under the compressive load and burst the face of the column. The bars are held in by means of lateral binding or hooping of light wire (not less than $\frac{3}{16}$ in. in diameter), which is regularly spaced throughout the height of the column. In circular columns the binding is often wound spirally round the vertical bars.

The distance between any two layers of binding is known as the "pitch"; it should not exceed $0{\cdot}6d$, where d is the smallest effective diameter of the column. For a distance equal to $1\frac{1}{2}d$ from the ends of the column the pitch should be equal to about half that in the body of the column.

The total *volume* of the binding should not be less than 0·5 % of the volume of the concrete contained within the hooping—generally known as the "hooped core".

It can be shown experimentally that the effect of the binding is to increase the total ultimate load which a column will carry, and the greater the amount of the binding the more marked will be this effect. The binding appears to aid the concrete in resisting failure by expansion and shearing due to compressive stress. On this basis it is usual to allow an increase in the maximum stress (c) which can be put upon the concrete. The theoretical treatment is beyond this volume, but the reader may consult the L.C.C. Reinforced Concrete Regulations to obtain some idea of the allowances which can be made.

221. Long columns in reinforced concrete. The above statement concerning short columns may be taken to refer to those with fixed ends in which the length (l) is not more than 15 times the smallest effective diameter (d).

For columns in which l/d exceeds 15, reductions in the maximum compressive stress (c) must be made in accordance with some theory

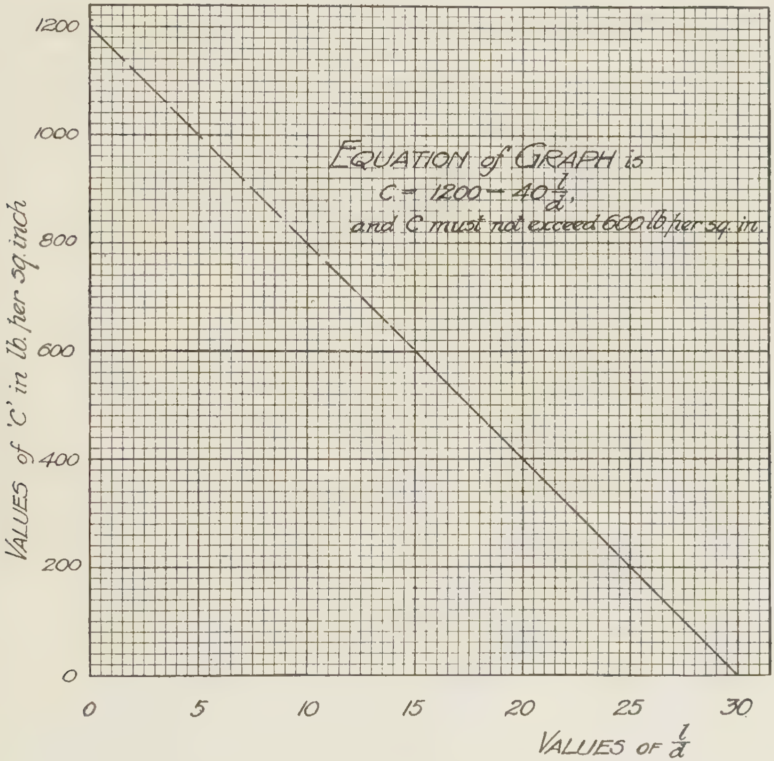


Fig. 275. Allowable stresses in reinforced concrete columns.

of the strength of long concrete columns. Not much experimental work in this direction has been done and the subject is not one that lends itself to simple treatment. For our present purpose, therefore, we may make use of the graph in Fig. 275, based upon the L.C.C. Regulations, which indicates the reductions to be made in c for various values of l/d.

As the graph indicates, the full value of 600 lbs. per sq. in. may be used on columns having a ratio of (l/d) up to 15, while beyond

$l/d = 25$ the allowable stresses become so low as to make it unprofitable to employ very slender columns of this type.

It should be noted that l in the examples so far dealt with refers to columns fixed at both ends. For any other type of end-fixing the *virtual*—or *equivalent*—length must be used as explained in para. 210 (see the following Example).

Example. *A reinforced concrete column which is* 14 *ft. in length has the section shown in Fig.* 274 *and may be assumed to be fixed at one end but only position-fixed at the other. Calculate the total safe load which may be put upon it.*

If the column had been fixed at both ends we could have reckoned its length as 14 ft.; this length must, however, be increased in the proportion corresponding to that existing between the "equivalent length" of a fixed column and a column having the fixing mentioned in this example, that is as between $l/2$ and $2l/3$; see para. 210 and Fig. 270. From this it will be seen that the "equivalent length" in this case

$$= \frac{14 \times \frac{2}{3}}{\frac{1}{2}} = \frac{14 \times 4}{3} = 18 \text{ ft. } 8 \text{ ins. or } 224 \text{ ins.}$$

Then $$l/d = 224/9 = 25.$$

Using a safe stress from Fig. 275 of 200 lbs. per sq. in. we have, since $A_e = 106$ sq. ins.,

$$\text{Total safe load} = 200 \times A_e = 200 \times 106 = 21{,}200 \text{ lbs.}$$

Experiment. If a series of test-columns can be made up, they should include (a) a plain concrete column, (b) a column with only vertical reinforcement, (c) a column with a small amount of binding, and (d) a column with a large amount of binding. Except for column (a), the ends should be enlarged to give an area equal to three or four times that of the section of the column, so as to prevent failure by crushing near the ends. Ample length should also be allowed between the enlarged ends and the column proper to allow the vertical reinforcement to take up its full load. A comparison between the load at failure and that for which each column is designed may be made as in para. 185, tests on beams.

222. Eccentric loads on reinforced concrete columns. The treatment of eccentric loads on reinforced concrete columns is not easy but, unless great refinement in design is required or very abnormal cases have to be dealt with, the approximate methods embodied in the following Example will be found to be satisfactory. Cases of eccentric loads arise frequently in columns—most practical columns carry eccentric loads—and in wall and arch members carrying lateral loads as well as direct thrusts.

The methods explained below are practically the same as those explained in para. 218 for long and short columns in homogeneous materials, the only difference arising in the calculation of the bending stresses in the column. We will limit the treatment to cases of symmetrical columns in which tension is not induced in the column and, for the sake of simplicity, we will include the "cover"

concrete in calculating the strength of the column.* Useful comparisons may be made with the treatment of doubly-reinforced beams given in para. 179.

Example. *Eccentrically loaded short column in reinforced concrete. The section of the column is shown in Fig.* 276. *It is assumed to be subjected to an eccentric load* (P) *of* 30,000 *lbs. acting* $1\frac{1}{2}$ *ins. away from the* Y–Y *axis. Find the maximum and minimum stresses in the concrete and steel.*

Equivalent section. In cases where no tension is developed in the concrete we may include in the "equivalent section" the whole area of concrete, as well as both portions of the reinforcement (compare with the case of a doubly reinforced beam and see Fig. 236). Since the reinforcement is symmetrically arranged the neutral axis $Y–Y$ will bisect the section. The equivalent sectional area may be taken to be the original area increased by two portions, arranged parallel to $Y–Y$, each equal to $(m-1)$ times the area of two bars; see Fig. 276. Then, since A_v = area of vertical reinforcement $= 4 \times 0{\cdot}442 = 1{\cdot}768$ sq. ins., the equivalent area

$$= A_e = 12 \times 10 + (m-1)\,A_v$$
$$= 120 + (14 \times 1{\cdot}768)$$
$$= 144{\cdot}7 \text{ sq. ins.}$$

Similarly the equivalent moment of inertia about $Y–Y$ (see para. 177 (A))

$$= I_e = \frac{10 \times 12^3}{12} + 2\,(m-1)\,(0{\cdot}884 \times 4{\cdot}125^2)$$
$$= 1861 \text{ inch units}^4.$$

Then the section modulus about $Y–Y$

$$= Z_e = \frac{I}{y} = \frac{1861}{6} = 310 \text{ inch units}^3.$$

And, using expression (i) from para. 218, we have

Maximum compression

$$= \frac{P}{A_e} + \frac{Pe}{Z_e} = \frac{30{,}000}{144{\cdot}7} + \frac{30{,}000 \times 1{\cdot}5}{310}$$
$$= 207 + 145 = 352 \text{ lbs. per sq. in.}$$

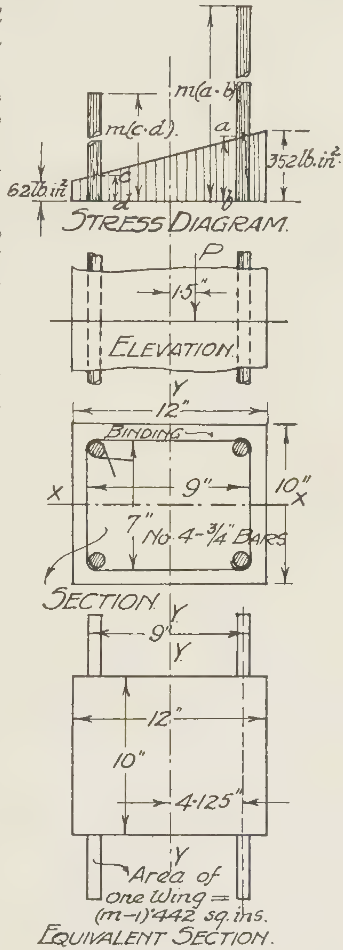


Fig. 276. Eccentrically loaded reinforced concrete column.

Similarly Minimum compression $= 207 - 145 = 62$ lbs. per sq. in.

* Many specialists insist on the inclusion of the "cover" concrete in the calculations, as being useful and active material.

The stress in the steel may be obtained from the stress diagram given in Fig. 276, by noting that it will be m times the stress in the concrete in the same layer.

Tensile stress in the concrete. It should be noted that when Pe/Z *equals* P/A in the above expression, then the stress at the edge farthest from the load will be zero. When Pe/Z *exceeds* P/A, then there will be tensile stress at this edge. A small tensile stress—say up to 60 lbs. per sq. in.—may in some cases be allowed. In cases of greater tensile stress, the reinforcement at this edge should be assumed to take the whole of the tension, and the effect of the concrete ignored between the neutral axis and this edge. The stress diagram is then similar to that for a doubly-reinforced beam. Owing to the fact that in such cases the position of the neutral axis varies with the magnitude of the eccentricity of the load P, the investigation is a troublesome one and is too lengthy to admit of inclusion in this volume.

Long columns with eccentric loads. These may be dealt with in the same manner, except that the maximum compressive stress allowed must be that shown in Fig. 275 for the corresponding value of l/d.

223. The effect of stiff connections between beams and columns. Before leaving this outline treatment of eccentric loads on columns it should be pointed out that, in practice, it is usually much more difficult to decide upon *the magnitude of the eccentricity* of a load on a column, than it is to design a column to carry such a load. This is in the main due to the varied nature of the connections transferring loads to the columns, which make it very difficult to decide the magnitude and nature of the stresses—usually bending stresses—imposed upon the columns through these joints. The nature of these difficulties may be best realised by considering a few typical connections.

At (A) in Fig. 277 a beam is shown supported upon a simple form of bracket. This is the form of connection which we assumed was used in Example 2 given in para. 218; the reaction of the beam may be taken to act at a distance of about half the projection of the bracket from the face of the stanchion, so that we can thus ascertain the magnitude of e, the total eccentricity. This connection would only be used for comparatively small beams.

At (B) a connection is shown which has, in addition to the bottom bracket, a cleat on the upper side of the beam. According to strength of this cleat, so will there be more or less of the end bending moment of the beam transmitted to the column. This will in effect increase the eccentricity.

At (*C*) a web cleat has been added, as for a deep beam, this makes a very stiff connection and, while it may not be able to transmit the whole of the end-bending moment (M) of the beam to

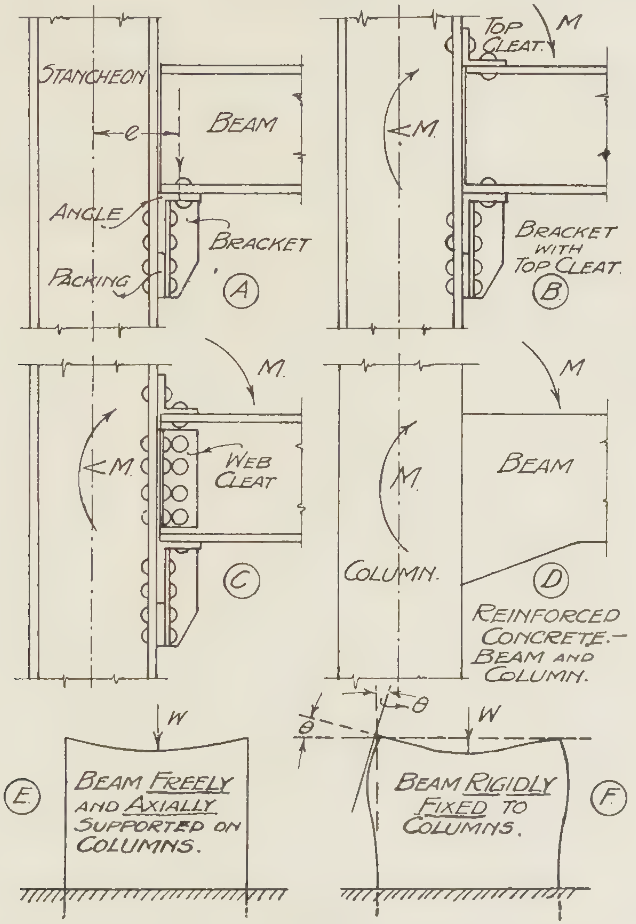


Fig. 277. Beam and column connections.

the stanchion, it will transmit a greater portion of it than would be the case in (*B*).

At (*D*) we have the connection between a reinforced beam and column. Such a connection, if adequately reinforced, would generally be considered to be quite rigid and to be capable of transmitting the whole of the end-bending moment of the beam to the column.

We therefore see that all columns will in practice lie between the two extremes indicated in the sketches (E) and (F), Fig. 277. In the first a beam is freely supported exactly over the axis of the column and transmits only a direct thrust to it. In (F) the connection is rigid so that the beam and column bend together, the inclination of the column to the vertical corresponding to the slope of the beam as shown. The theoretical investigation of these effects is beyond the scope of this volume.*

If some form of cross-bracing cannot be used, stiff connections between beams and columns are of value in increasing the rigidity of a building as a whole, thus helping it to resist side thrusts, such as those due to wind pressure.

Bases of Columns. Grillages. Foundations

224. (a) **Steel column base and grillage.** The load from a steel column or stanchion is usually transmitted to a base plate, this plate being of such dimensions as to reduce the stress on the supporting material (stone or concrete) to a safe value.

Vertical side plates -known as "gusset plates"—are riveted to the flanges of the stanchion and are used so as to spread the pressure over a sufficient width of base plate. The base plate and the gussets are connected by means of angles, countersunk rivets being used on the undersurface of the base plate.

In connecting the gussets to the flanges and the angles to the gussets, sufficient rivets of a suitable diameter (d) are used to transmit the whole of the flange-load (or a specified portion of it) to the base plate. Web angles are similarly used to transmit the load carried by the web. The rivets are calculated for shear or bearing, noting that the gusset rivets are in single shear, while the web rivets are in double shear. The gusset plates should therefore be at least $0{\cdot}4d$ in thickness; see para. 162.

If a large load has to be transmitted to the ground through the foundations of a column, the foundation block may project a considerable distance beyond the edge of the base plate; see Fig. 278. In such a case either a very deep foundation block of concrete is used—so that lines drawn at an angle of not less than 45° from the edges of the base plate are wholly contained within the block (see para. 206)—or a "grillage" is formed of two or more layers of steel joists encased in concrete.

Such a grillage, designed for a column carrying a load of 100 tons, is shown in Fig. 278. The foundation block is 8 ft. 6 ins. by 8 ft. 6 ins., thus reducing the pressure on the ground to $1\frac{1}{2}$ tons

* For a treatment of monolithic beams and columns in reinforced concrete the reader is referred to Hudson's *Reinforced Concrete*, or to Faber and Bowie's *Reinforced Concrete Design*.

per sq. ft. The joists in such a grillage may be designed as double cantilevers, on the assumption that they bear a uniform upthrust on their undersurfaces, and a uniform downward pressure of a greater intensity from the steelwork transmitting the column load.

Fig. 278 (A) shows how the bottom tier of joists receives its load from the upper tier, while Fig. 278 (B) gives the load and space

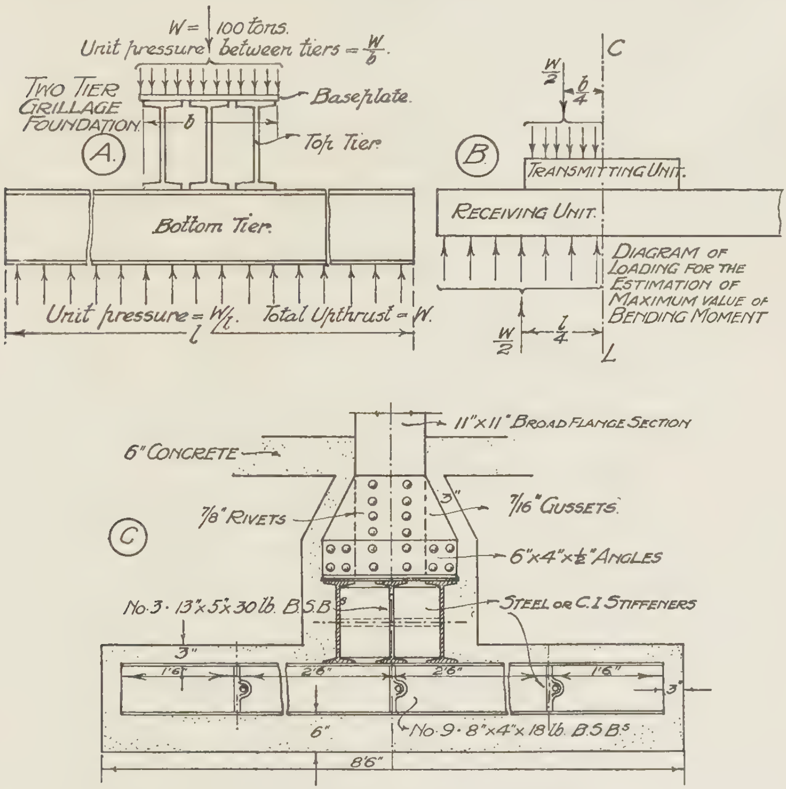


Fig. 278. Base and grillage to steel stanchion.

diagram from which it is possible to estimate the magnitude of the bending moments, which can be shown to reach a maximum at the centre line C–L.

Thus, taking moments about C–L to the left, due to half the load acting downwards, together with half the supporting forces acting upwards, we have

$$B_{C-L} = B_{\max.} = \left(\frac{W}{2} \times \frac{l}{4}\right) - \left(\frac{W}{2} \times \frac{b}{4}\right) = \frac{W}{8}(l - b). \quad \ldots\ldots\text{(vi)}$$

This expression will give the value of $B_{\max.}$ for any slab, or tier of joists (of length l), receiving load from another slab, or tier of joists, or base plate (of breadth b), but it applies only to symmetrically arranged cantilever foundations. The bending moment so determined is resisted by the full breadth of the slab, or by the whole tier of joists. (The foundation shown in Fig. 278 has been designed on the above principles and should be checked by the reader as an exercise; see Problems XXIV.)

In order to ensure that a tier of joists acts effectively as one unit, the upper joists are bolted together into a series with steel

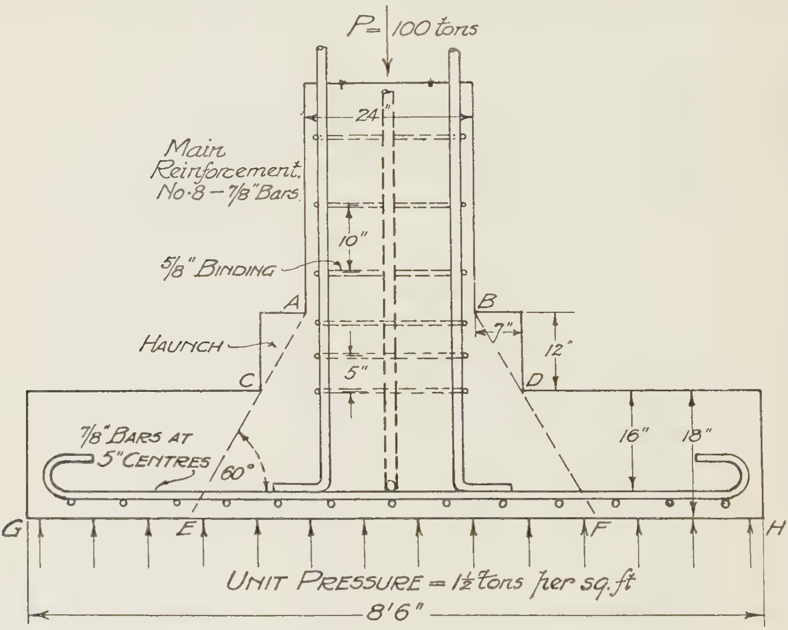


Fig. 279. Base of reinforced concrete column.

or cast iron packings fitted between the joists to form rigid connections. The lower series may be treated similarly, or connected by channels bolted to the upper flanges. The whole is then encased in concrete.

(b) **Foundation block for a reinforced concrete column.** Details of a column and foundation block in reinforced concrete, carrying the same load and having the same foundation pressure, is shown in Fig. 279. The column is 24 ins. square.

It is assumed in this case that the concrete of the foundation block can carry the compressive load within the sloping lines drawn

at 60° from the outer edges of the column, see lines AE and BF in Fig. 279. (See notes in para. 206 on the compressive strength of concrete.) The upper part of the block may thus consist of a plain concrete portion 12 ins. deep, the points C and D lying on the 60° lines. We might then consider the portion of the lower block which is beyond the points E and F as being subject to bending as in a cantilever; it is safer and more usual, however, to calculate the bending moment at the vertical section passing through C (and D) and due to upthrust acting on the projection. This portion of the block is then designed as a simple rectangular cantilever, the depth of the lower portion of the block and the amount of reinforcement being determined in the usual way, as for a beam. The reinforcing bars are placed at right angles to each other in two exactly similar layers uniformly spaced. The total depth of the foundation block should be checked to ensure that there is a sufficient length of the column bars embedded within the block to enable them to take up their full load—which, it should be remembered, may only reach 9000 (when $m = 15$) and not 16,000 lbs. per sq. in. as in a beam; see para. 180 on Adhesion.

The reader should work through the usual calculations and seek to justify the sizes given in Fig. 279. Other methods of arranging the reinforcing bars are of course in practical use, and further refinements in design are also possible.

Problems XXIV

1. A short cast iron column of hollow-circular section, 6 ins. outside diameter and $4\frac{1}{2}$ ins. inside section, is subjected to an eccentric load of 18 tons, which acts at 2 ins. from the centre of the section. Find the stresses induced in the extreme fibres of the column.

2. Using the rule set out in statement II in para. 218, find the safe load which may be put upon a cast iron column of the same section as that described in Prob. 1, but being 10 ft. long, the load having an eccentricity of 1·25 ins. The column may be assumed to be flat-ended.

3. If as is suggested in Example 1, para. 218, the principal rafter is made up from two angles 5 ins. by $3\frac{1}{2}$ ins. by $\frac{3}{8}$ in., calculate the safe load which may be put upon the rafter, the rivets lying on a line 3 ins. from the lower edge of the rafter, and the ends being assumed to be free ends, see Fig. 273 (i).

4. Calculate the maximum and minimum stresses in the column section shown in Fig. 273 (ii), if the loads are: A, 6 tons; B, 8 tons, and C, 6 tons.

5. Find the equivalent area of the rectangular column section shown in Fig. 274 if $D_1 = 14$ ins., $D_2 = 20$ ins., the cover of concrete is 2 ins. all round, and each of the six vertical bars are 1 in. in diameter. What total load would this section carry if the safe stress on the concrete is to be 500 lbs. per sq. in.? (The "cover" concrete may be included in calculating the strength of the column.) What would be the stress in the steel?

6. A reinforced concrete column of the section shown in Fig. 276 has a length of 13 ft.; calculate the safe total load which may be placed upon it if the ends may be assumed to be fixed ends.

7. If a concrete member of the rectangular section shown in Fig. 274, and dimensioned as in Prob. 5, is used to form a portion of a three-pinned reinforced concrete arch, find (*a*) the maximum eccentricity of the load at any section which will just reduce the pressure at one edge of the section to zero, (*b*) the maximum load which may be carried by the section with this eccentricity (the neutral axis of the section coinciding with the middle reinforcing bars, and the stress in the concrete not exceeding 500 lbs. per sq. in.), and (*c*) the maximum stress in the steel at this load.

8. If a grillage foundation is to be designed having the same bottom area, projection, and number of beams in the bottom tier as is shown in Fig. 278, but carrying a load of only 80 tons, calculate suitable sections for the beams.

9. Calculate suitable reinforcement for the foundation block in Fig. 279, if the total load remains at 100 tons, but the lower block is made 10 ft. square and 2 ft. thick. The upper block of concrete may be kept of the dimensions shown in Fig. 279. The bottom reinforcement should be calculated for the conditions existing at sections corresponding to those at C and D in Fig. 279.

CHAPTER XXV

THE STRENGTH OF TIMBER. TIMBER CONSTRUCTION.

225. The strength of timber. It would not be possible in the space at our disposal to discuss the variations in strength and elasticity which characterise the many varieties of timber now in general use. For this reason, and also in the interests of simplicity, we shall limit our discussion to a few typical timbers from among those which are definitely used in this country for structural purposes. The timbers which we shall select for this purpose are: (I) Pitch Pine (*Pinus palustris*, etc.), (II) the more regular grained Oaks (*Quercus*, various), (III) Northern Pine or Scots Pine (*Pinus sylvestris*), and (IV) White Deal or Spruce (*Picea excelsa*). Such a limitation will be found to reduce considerably the difficulties which are commonly associated with the testing of timber, and which arise largely from the great variety of woods available for use. If other timbers are to be used which cannot be grouped readily with those mentioned above, it ought not to be difficult, after reading the following notes on testing, to obtain the necessary ultimate and the working stresses, and to apply them in the design of timber structures.*

The testing and grading of structural timbers. Apart from some excellent work which was done during the war in connection with the construction of aircraft, there has not been, up to the present, any general attempt in this country either to apply methodical tests to timber, or to grade it according to its structural uses. This has no doubt arisen in part from the fact that the design of timber structures on the basis of correct mechanical principles, has so far received little attention in this country, in spite of the fact that the adoption of more scientific methods of design would certainly result in the production of more efficient and possibly less costly structures. Such work would be greatly assisted by the issue of timber specifications and constructional regulations more in touch with our present-day knowledge of timber construction than are those at present in use.

In this connection, it is anticipated that the work now being done at the Forest Products Research Laboratory at Princes

* For a fuller treatment of this subject readers may consult J. B. Johnson's *Materials of Construction*, Unwin's *Testing of Materials of Construction*, or *The Properties of Engineering Materials*, by Popplewell and Carrington.

Risborough will eventually lead to the preparation of specifications for the grading of structural timbers.*

Experiment. The testing of timber.

(A) **Compression.** It will be convenient to limit our consideration of the compressive strength of timber to those cases where it is subjected to a load (*a*) in a direction parallel to the grain or, as we shall call it, **end grain compression**, and (*b*) in a direction at right angles to the grain or, as we shall call it, **side grain compression.**

(a) **End grain compression.** A convenient form of test piece is shown in Fig. 280. This is square in section and of a length equal to four times the lateral dimension. As a rule compression failure is accompanied by a crushing or crumpling of the fibres as shown. Average values for the ultimate crushing strength of ordinary specimens of the timbers named above are: Pitch Pine 6000 lbs. per sq. in., Oak 6000 lbs. per sq. in., Northern Pine 5000 lbs. per sq. in., White Deal 4500 lbs. per sq. in.

To obtain safe or working stresses these values are divided by a factor of safety; suitable values are given in Table XV in this chapter. (**Note.** We shall use the small letter c to refer to the safe end grain compressive stress.)

(b) **Side grain compression.** This is sometimes referred to as the "bearing" strength of timber, in reference to the fact that it is the stress called into play when a beam rests or "bears" upon its supporting surfaces at each end. To carry out this test accurately the load may be applied to the specimen through a steel plate, as shown in Fig. 280, when measurements of the strain may be made as the test proceeds. A simpler method of carrying out this test, and one in which the conditions approach those to be met with in practice, is also indicated in Fig. 280. A rectangular block of the wood to be tested is placed upon the table of the testing machine with its fibres running horizontally, on this is placed another block, either of the same or a harder wood, with its fibres vertical and the whole subjected to a gradually increasing load. When the load has reached a certain figure (the *elastic limit*) a slight indentation will be noticed on the upper surface of the lower block when the load is removed. With only slight increases in the load, this depression will go on increasing to a considerable depth. It is generally assumed that the *ultimate strength* has been reached when the depression amounts to about 10 or 12 % of the original thickness of the lower block. Average values for the

* On this point and for further details of the testing of timber, readers may consult the publications of the Forest Products Research and of similar organisations in other countries.

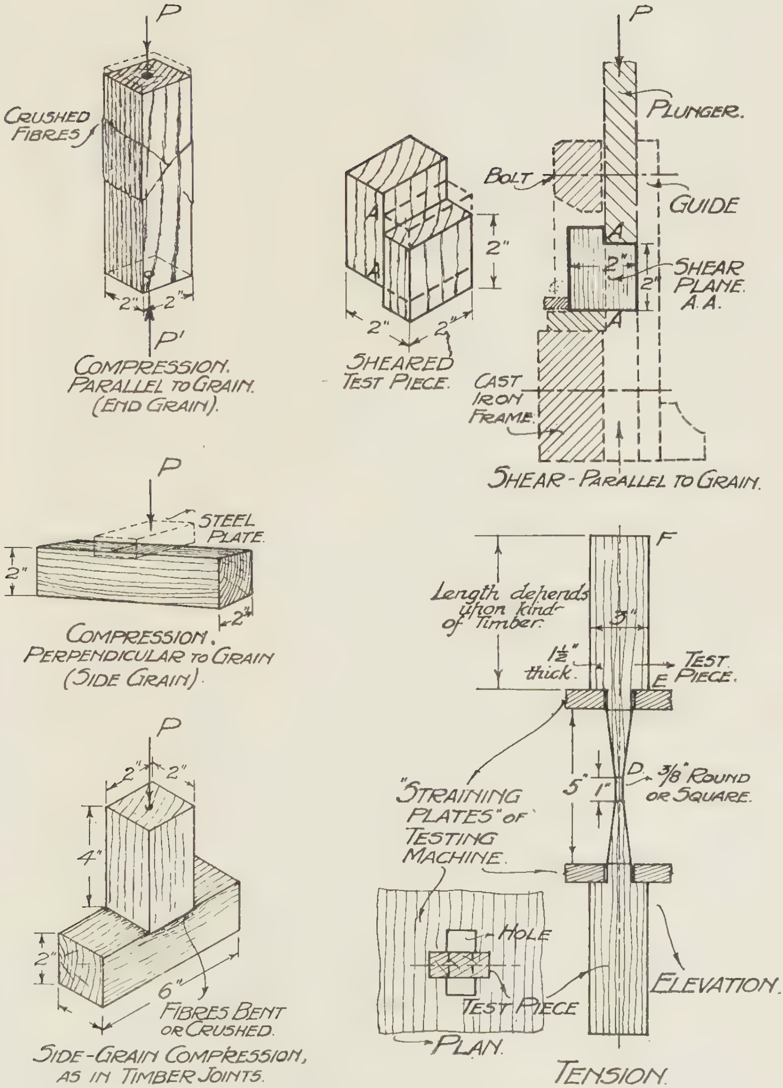


Fig. 280. Test pieces in timber.

selected timbers are: Pitch Pine 650 lbs. per sq. in., Oak 1000, Northern Pine 550 and White Deal 450.

To obtain the working stresses -which we will indicate by c_s, compression on *side* grain—these figures are divided by a factor of safety of 2 to 3. Such a low factor can only be justified on the grounds that the elastic limit and ultimate strength figures are close together, so that the factor is practically used to divide the elastic limit stress. If, however, live loads have to be borne in this way, then the stresses adopted must have a still lower value.

It will be noted that the safe side-grain compressive stresses are only about one-quarter of those for end-grain compression. When it is remembered that the wood structure is made up, in the main, of tube-like cells which lie in the direction taken by the grain of the wood, then this marked difference in strength will not be difficult to account for, since any form of thin tube will be much more readily crushed by lateral pressure than by longitudinal pressure. This fact is one which must be borne in mind since it is of vital importance in the design of timber joints.

(B) **Shear—Longitudinal shear or shear along the grain.** The test to ascertain the longitudinal shear strength of timber is not an easy one to carry out. Special grips have to be used and it is difficult to avoid some amount of bending strain on the specimen. For the comparative tests which we are here considering, a grip of the type shown in Fig. 280 may be used, in which the load is transmitted to the specimen through a plunger working in guides, *A–A* being the surface to be sheared.

Average values for the selected timbers are: Pitch Pine 700 lbs. per sq. in., Oak 900, Northern Pine 400 and White Deal 300. Working stresses—which we will indicate by the letter s—are given in Table XV.

Longitudinal shear in bending. The relatively low values for longitudinal shear is another characteristic of the strength of timber which greatly influences the design of timber joints. It is also a vital factor in the design of timber beams, particularly when they are deep in comparison with their length.

If a series of tests to fracture be carried out on wood beams having a depth equal to at least one-eighth of the span and centrally loaded, then the majority will probably fail by horizontal shear near the ends. In these cases if the maximum shear stress at failure be calculated (see Chap. xv), it will generally be found to be lower than that obtained from the direct shear tests described above. This is due usually to the presence of unsuspected defects in the ends of the beam, which would have been more easily detected in the smaller shear test pieces. For similar reasons it is usual to

adopt a lower working shear stress in the case of beams, which we will indicate by s_b, safe shear stress in *bending*. Suitable values are given in Table XV.

Table XV

Maximum working stresses in Timber (lbs. per sq. in.)

Typical timber	Compression		Tension	Shear		Modulus of elasticity (E)
	End grain (c)	Side grain (c_s)		Along grain (s)	Bending (s_b)	
I. Pitch Pine (*Pinus palustris*)	1200 (1000*)	300	Same as for "End grain Compression" (c)	180	120	1,500,000
II. Oak (*Quercus*, various)	1200 (1000*)	450		200	130	1,200,000
III. Northern Pine or Scots Pine (*Pinus sylvestris*)	1100 (950*)	250		120	100	1,200,000
IV. White Deal, Spruce (*Picea excelsa*)	900 (750*)	225		100	80	1,200,000

* Limiting stress for columns, see para. 227.

(C) **Tension.** In comparison with the other values which we have considered, the strength of timber in tension, in a direction parallel to the grain, is remarkably high, being usually two or three times that for compression in the same direction. It is in fact so high relatively, that in practice it is found almost impossible to take full advantage of it, the connections at the ends of timber tension members always breaking down before the full strength is developed. The same difficulty is found in carrying out tension tests, since in the ordinary testing machine grips the ends of the test piece are crushed and fail to transmit the necessary load. A form of test piece which has been developed by the authors and which overcomes some of the difficulties mentioned is shown in Fig. 280. The load is transmitted to the specimen through carefully cut shoulders which bear upon the straining plates of the testing machine. The centre of the test piece is reduced to a small section, either square or cylindrical. The lengths DF and EF must be of such dimensions as to prevent failure of the test piece by longitudinal shear. The small sketch shows in plan how the test piece is passed through a rectangular hole in the plates and then turned so that the shoulders take a bearing on the plates. Though tension tests are usually omitted, it will be found to be very instructive to carry out a few comparative tests in the manner here described.

It will be both convenient and in accordance with normal practice to adopt the same working stresses in tension as are used in compression; see Table XV.

(D) **The modulus of elasticity** (E). The value of E for timber may be obtained directly by strain-measurements on a specimen in compression. A simpler method is to obtain it from deflection measurements on loaded beams. Rather more reliable values for E can be obtained by using two-point loading on the beam instead of central loading; such an experiment is described in para. 143. Average values for E are given in Table XV.

226. Factors affecting the strength of timber.

Moisture. In this country, timber generally reaches the user in a seasoned condition, and is used in positions where the changes in the moisture-contents of the timber are mainly those due to the daily variations in humidity (see Vol. I for a method of determining the moisture-content of timber). Under ordinary conditions in this country, the amount of moisture contained in air-dried timber ranges from 10 to 18 %. In the majority of timber tests the figure of 12 % is adopted. While a smaller amount of moisture than 12 % will bring an increase in strength, it may be accompanied by an increase in brittleness. On the other hand an increase in moisture means a definite reduction in strength, which, with the class of timber we are discussing here, continues until a moisture-content of about 25 % is reached, from which point the strength remains fairly constant with further increase in moisture. If a timber structure is to be erected in a position in which it may be continuously wet, or be alternately wet and dry, then some reduction—up to 25 %—must be made in the working stresses, since the strength of timber varies considerably with the amount of moisture which the timber contains.

Defects in timber. Apart from the defects due to faulty seasoning or incipient decay—which need not be discussed here—the defects to be avoided in the members of timber structures are knots, shakes and cross-grain. It is not intended to suggest that such defects cannot be allowed in such timbers, since it would be impossible to satisfy such a requirement, but these defects if present must not be of such a nature, or so placed, as to affect seriously the efficiency of the member. Thus, while large and loose knots ought not to be countenanced under any circumstances, small and well-distributed knots, *which do not seriously disturb the regular lie of the grain*, need not be looked upon as a source of weakness, unless they occur in an area subject to tensile stress. Again, while large longitudinal shakes near the end of a short beam could not be

allowed to pass, the presence of small shakes in other members not subjected to high shear stress need not occasion alarm. Cross-grain or, as it is sometimes called, "short grain" should not be allowed in any important structural member.

Timber Columns

227. It will be convenient, before proceeding to the more general consideration of timber construction, to complete our consideration of the subject of columns by an investigation of the strength of timber columns.

The age-long use of the familiar scaffold pole may be looked up as a practical recognition of the fact, that timber possesses to a remarkable degree those qualities of strength and flexibility which go to the making of a good "column". As we have seen, however, timber is not a material which, like some of the metals, is capable of resisting intense lateral stresses; hence any attempt to "fix" the ends of timber columns can only be moderately successful, and it is thus difficult to take full advantage of the strength of the timber in the direction of the length of the column. Much may, however, be accomplished by careful construction, as we shall attempt to show in this chapter.

Column formulae for timber. (a) **Johnson-Euler formulae.** As several experimenters have been able to show, the buckling loads obtained from Johnson-Euler formulae (constructed in the manner set out in Chap. XXIII) agree remarkably well with the experimental results obtained from large scale tests on timber columns. If it is desired to use this type of formula, the reader should be able, from the explanation given in Chap. XXIII and from the strength-figures given for timber in para. 225 and in Table XV, to ascertain the magnitude of the various constants. As indicating the magnitude of the values to be expected, the following formulae for Northern Pine columns may be set out.

Johnson-Euler formulae; Northern Pine:

Free ends. P/A = buckling stress $= 4000 - 28\,(l/g)$, ...(i)

Flat ends. $P/A = 4000 - 18\,(l/g)$. ......(ii)

Using the working stress of 1100 given in Table XV we obtain a working factor of (4000/1100) or 3·65. If desired a higher factor may be used in the case of the more slender columns.

(b) **Simplified straight-line formulae.** In view of the fact that timber columns are usually of simple section—square, rectangular or circular—a type of column formula may be devised in which d, the least diameter or least lateral dimension, is used instead of g. Such formulae, which are widely used in America in designing

timber structures, are usually arranged to give the safe working stress. A common form is

Free ends.

$$\frac{P}{FA} = p = \text{safe working stress} = c\left(1 - \frac{l}{40d}\right). \quad \ldots\ldots\text{(iii)}$$

Flat ends.

$$\frac{P}{FA} = p = \text{safe working stress} = c\left(1 - \frac{l}{60d}\right). \quad \ldots\ldots\text{(iv)}$$

The value of c which is used in these expressions is that given in Table XV. It is usual, however, to indicate a lower limit above which the value of p must not rise. (See values given in brackets in Table XV, and line *def* in the graph of the formula in Fig. 281.) *In the case of circular sections the constants* 35 *and* 50 *should replace the constants* 40 *and* 60 *given in expressions* (iii) *and* (iv) *respectively.*

It will be obvious from the form of the above expressions that when the ratio l/d equals 40 and 60 respectively then the value of p will be zero. This has the effect of limiting the length of column which it will be profitable to use and is in the interests of safety; e.g. in the case of flat-ended rectangular columns the allowable stress at a ratio of $l/d = 45$ would be only 300 lbs. per sq. inch.

The full range of expressions for flat-ended rectangular columns for the typical timbers given in Table XV is as follows:

Flat Ends.

I. **Pitch Pine.** $p = 1200\ (1 - l/60d)$, but the value of p must not exceed 1000 lbs. per sq. in.

II. **Oak.** $p = 1200\ (1 - l/60d)$, but p must not exceed 1000 lbs. per sq. in.

III. **Northern Pine.** $p = 1100\ (1 - l/60d)$, but p must not exceed 950 lbs. per sq. in.

IV. **White Deal.** $p = 900\ (1 - l/60d)$, but p must not exceed 750 lbs. per sq. in. ......(v)

Example. *Draw graphs of the working stresses on flat and free-ended pitch pine columns, using the appropriate expressions given above. From these graphs find the safe load which can be put upon a pitch pine shore, 15 ft. long and 9 ins. by 9 ins. in section,* (a) *if the ends are assumed to be free, and* (b) *if they are assumed to be fixed or flat.*

The construction of the graphs based on these expressions is not difficult. Two lines are drawn from a, in Fig. 281 (stress of 1200 lbs. per sq. in.), to b, ($l/d = 60$), and c, ($l/d = 40$), respectively. A horizontal line is then drawn from d (the limiting stress of 1000) to meet the first two lines and the graphs are complete.

In this case $$\frac{l}{d} = \frac{15 \times 12}{9} = 20.$$

(a) **Free ends.** The safe stress from the graph is 600 lbs. per sq. in., hence

$$\text{Safe load on the column} = 600\,(9 \times 9)$$
$$= 48{,}600 \text{ lbs.}$$

(b) **Flat ends.** Safe stress = 800 lbs. per sq. in., hence

$$\text{Safe load} = 800\,(9 \times 9)$$
$$= 64{,}800 \text{ lbs.}$$

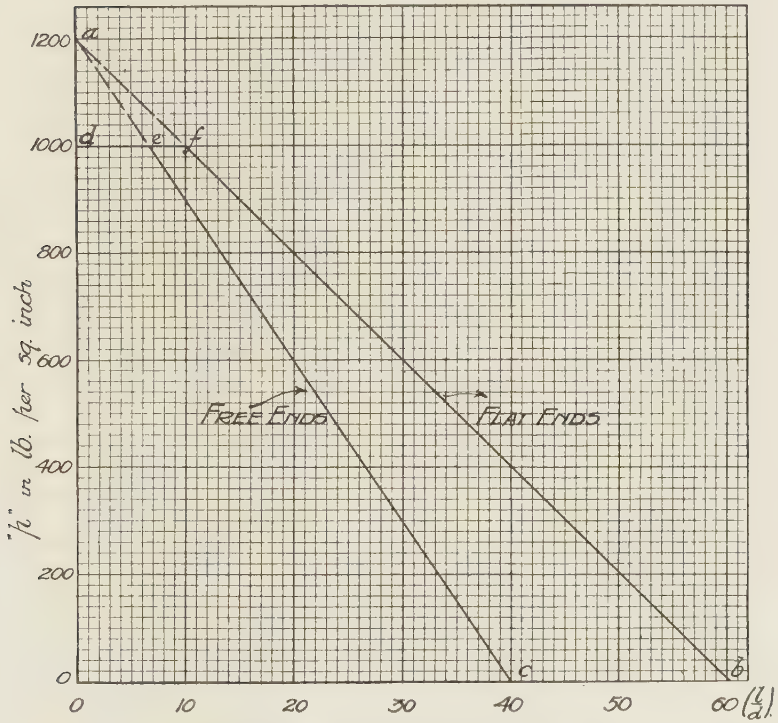


Fig. 281. Values of p for pitch pine columns.

228. The "End-fixing" of timber columns. It is not as a rule easy to classify timber columns, according to the manner in which the ends are secured, into "free", "flat" or "fixed" ends. To take the shore in the preceding example as a case in point, if each end rested on some material, metal or stone, which was at least capable of exerting the full stress of 800 lbs. per sq. in., then we might look upon the shore as a "flat-end" column. If, however, it is to rest or bear upon timber pieces laid on their sides, then we may not be able to develop even the stress of 600 lbs. per sq. in.

unless very hard timber is used for these pieces. In the latter case it would be safer to treat the column as having "free ends"; but see para. 230.

In the case of jointed structures the position is not much better, unless special joints can be devised which develop the full strength of the timber member as a column; see para. 234. Except in the latter case, timber compression members should be dealt with as free-ended columns.

In the majority of temporary structures, in which the members are relatively light and slender and connected to each other by simple means—bolts, screws or even nails—then the compression members should always be treated as having "free ends".*

THE DESIGN OF JOINTS AND CONNECTIONS IN TIMBER STRUCTURES

229. In considering the design of the various connections and joints which are used in timber structures, it must be understood that we shall deal here only with those connections which are designed to support and transmit relatively large forces, and where it is necessary that the amount of material used and the labour needed to construct the joint be kept as low as possible. We are thus not concerned with the many joints used in the construction of light framework which has not to carry much load beyond that due to its own weight.

In order to make our investigation more methodical a rough classification of the connections has been adopted which will, it is hoped, be sufficiently explained by the headings to each paragraph.

230. Joints in bearing—End to side grain. The most common type of connection in timber structures is that in which one timber carrying a load rests or presses against another lying at right angles to it; see Fig. 282 (A). In a simple case where a post merely rests upon a sill, it will be obvious from our foregoing discussion that *the load which can be carried by the post will be limited to the safe load which can be put on the side grain of the sill.* The *efficiency* of such a joint must therefore be low; for example if the pitch pine shore dealt with in the Example in para. 227 is to rest on a pitch pine sill, then the total safe load which can be put upon it is

$$\text{Safe load} = \text{bearing area} \times \text{side grain stress } (c_s)$$
$$= (9 \times 9)\ 300 = 24{,}300 \text{ lbs.},$$

which is only half that calculated for even a free-ended column.

* A very full discussion of this and similar problems relating to timber construction will be found in a paper read to the Institution of Structural Engineers by Mr G. A. Gardner, M.I.Struct.E., on "Some Aspects of Timber Construction". See the *Structural Engineer*, April 1925.

It follows, therefore, that we must if possible avoid any cutting which will reduce the bearing area. Thus if a tenon is to be used *merely to keep the post in position*, as is shown at (A), Fig. 282, then the tenon should not be larger than is necessary for this purpose. Any arrangement by which the post can be kept in position without reducing the bearing area will make for a more efficient and stronger connection.

If, as is usual, a shore is to take its load from a horizontal needle through a pair of wedges, as shown at (B), Fig. 282, then every care must be taken in cutting and driving those wedges, to

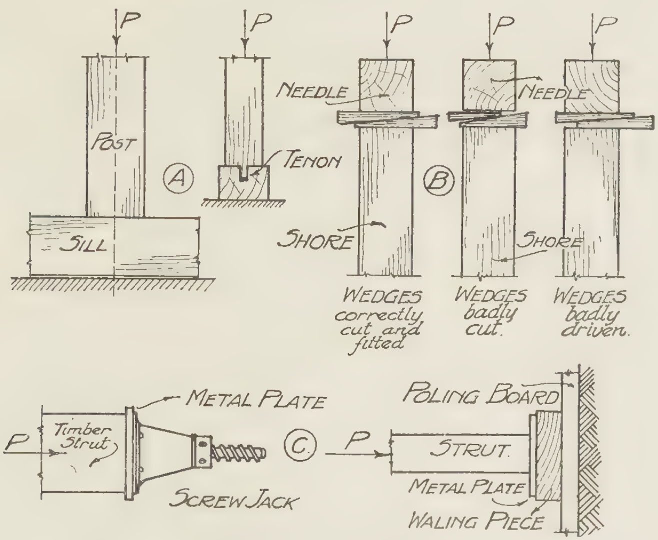


Fig. 282. Timber connections. Bearing—end to side grain.

see that the load is transmitted uniformly through the wedges to the bearing surface of the shore. When laid together the upper and lower surfaces of each pair of wedges should be exactly parallel (see sketch showing the effect of badly cut wedges), and each wedge should at least reach to the opposite side of the joint before driving tightly into position (see sketch showing the effect of bad driving, in which the load on the shore is only transmitted through part of its end area, producing the effect of an eccentric load).

The efficiency of such joints can be increased by using a stronger timber for the needles and sills, and also for the wedges (thus the use of an oak sill and needle with oak wedges would have increased the safe load on the pitch pine post by 50 %, the value of c_s for

oak being 450). Metal bearing plates can also be used, the area of the plate being proportioned to the strength of the weaker timber surface. For example, the timbering to the sides of trenches is usually supported by struts which bear upon horizontal members called "walings"; see Fig. 282 (C). The struts usually bear directly upon the walings, the force exerted being thus limited by the side-grain compressive strength of the timber—usually quite low. By inserting thick wrought iron or steel plates or wedges between the struts and the walings the full strength of the struts can be developed. The bearing area of the plates or wedges on the timber must of course be proportioned to the side-grain strength (c_s). Similarly if, as is now often done, a screw jack is used in conjunction with the struts, then for the same reason a metal plate should be interposed between the jack and the timber surfaces on which it bears; see Fig. 282 (C).

In the case of posts used to support concrete shuttering, their strength will be similarly limited to that of the weakest bearing surface, so that, if this surface be the underside of the boarding or of a cross bearer, it will only be possible to stress the posts up to the side-grain strength of the bearers, which will usually be much less than the post could carry, unless it is very slender. This is another case in which the use of metal bearing plates would permit of larger loads being carried.

231. Joints in bearing—End to end grain. When one post bears upon another, end to end, as shown in Fig. 283 (A), there need be no reduction in strength at the joint, except that due to the holes for bolts which are used to hold one post correctly upon the other. If such a joint occurs in a long column it should be situated where it can be secured laterally by stays or other horizontal members of the structure, otherwise a much more elaborate form of joint will be necessary (see para. 236). As is shown at (A) and (B), Fig. 283, the use of steel plates can be avoided by connecting the two parts of the post by means of a simple bolted half-lap joint.

In temporary structures posts are frequently built up from three equal sized pieces as shown in Fig. 283 (C), the pieces being held together by bolts. To secure the full value of the compressive strength of such a post, the butt joints should be accurately cut and fitted and arranged to "break joint" as shown in the figure. In the case of derrick towers these posts are supported at frequent intervals, hence the strength of the individual lengths of post may be calculated as for short columns. If possible joints in the posts should not occur between the points at which the posts are secured laterally.

In constructing timber buildings, the framework usually consists of vertical posts with horizontal sills at each floor level. Two alternative arrangements are shown at (D), Fig. 283. In the first the strength of both posts would be reduced by resting on horizontal members, in addition the posts would rise and fall with the moisture-movements of the two horizontal pieces and thus produce "seasonal cracks" within the building. It will therefore be seen to be structurally advantageous to adopt the second arrangement, keeping the posts continuous and jointing the horizontal members into them as required.

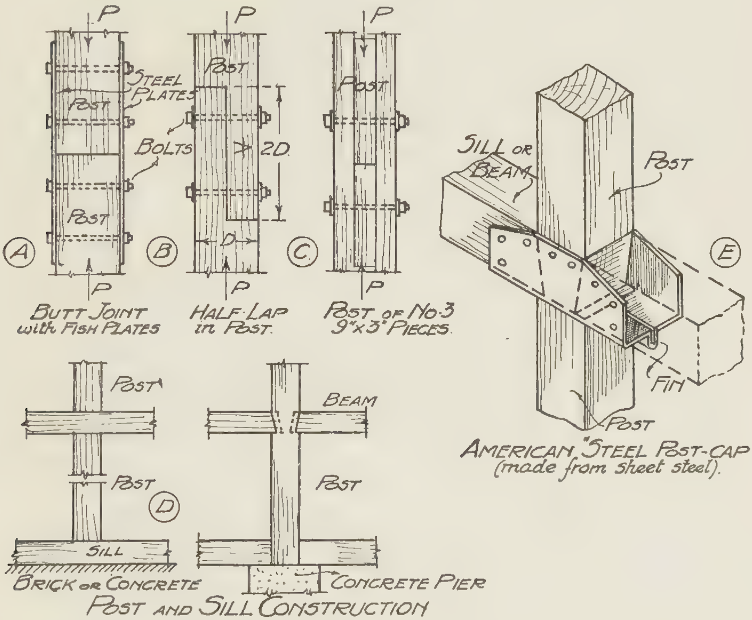


Fig. 283. Timber connections. Bearing—end to end grain.

Some difficulty is usually experienced in making a satisfactory connection between a beam, carrying a floor, and a post in this form of construction. If the beam is housed into the post, as in the second sketch at (D), Fig. 283, the strength of the post may be considerably reduced. In America this difficulty is overcome in an efficient and economical way by the use of post-caps, of which an example in sheet steel is shown in Fig. 283 (E). The cap, beams and post are held in position by the "fin", which is let into the top of the lower post, and by a few screws through the side of the cap. All the cuts are "square cuts", so that the connection is both economical and efficient.

232. Compressive strength of timber on inclined surfaces. We have already ascertained by experiment the stresses which can be

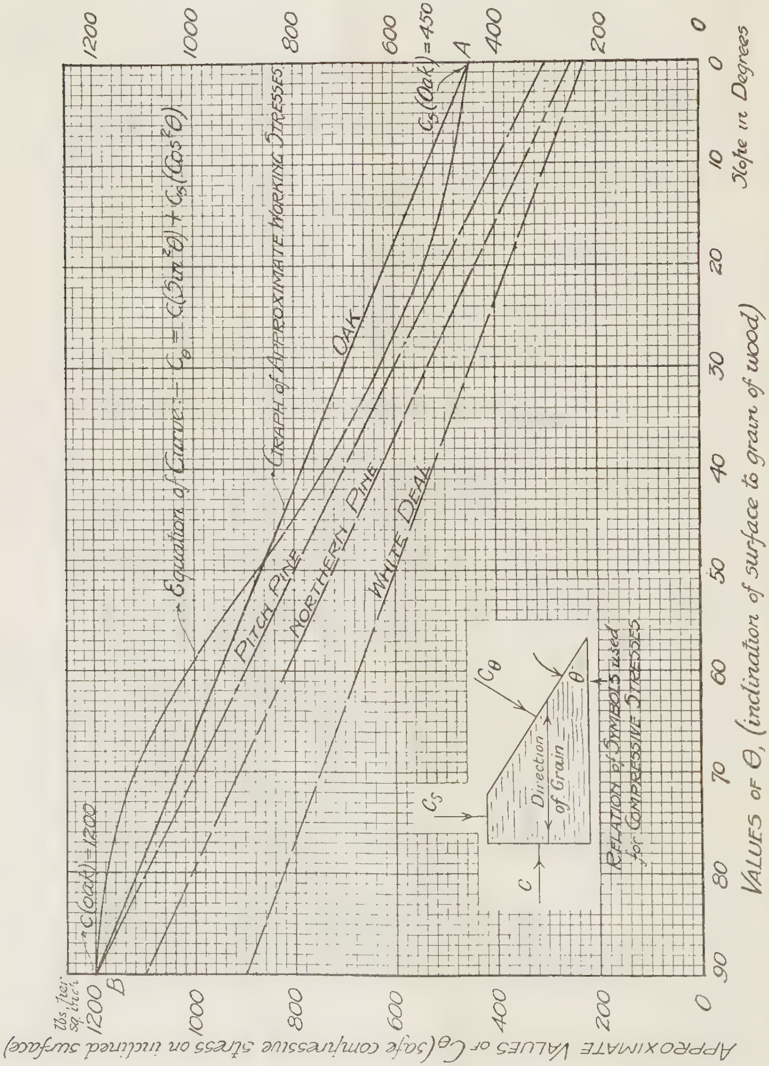


Fig. 284. Approximate values for compressive stress on inclined surfaces of timber.

put upon timber on surfaces which are (*a*) at right angles to the grain (end grain stress), and (*b*) parallel to the grain (side grain stress). Before we can deal with joints having surfaces at other

inclinations we must be in a position to state what safe stresses these inclined surfaces will carry.

If c_θ be the safe compressive stress which can be put upon a surface *inclined at an angle* θ *to the lie of the grain*, and c and c_s are the safe end grain and side grain stresses respectively, then it has been shown by Jacoby* that c_θ varies in accordance with the following expression

$$c_\theta = c\,(\sin^2\theta) + c_s\,(\cos^2\theta). \qquad \text{......(vi)}$$

The graph of this expression in the case of oak is given in Fig. 284 by the curved line from B to A.

A simpler straight-line graph, giving approximate values which the authors suggest will be found to be sufficiently accurate for practical purposes, may be obtained by joining the c_s value at 0° to the c value at 90°; see Fig. 284. The straight line AB gives this graph for oak. The graphs for the four typical timbers have been drawn in this way and will be used in the calculations set out below.

233. Joints to resist inclined thrust. Nearly all the more elaborate timber joints come under this head. The variety of form which they have shown in the past has been immense. Their consideration from the point of view of strength may, as we shall see, tend to simplify the forms which they should take.

Consider first the case of a strut inclined to a vertical post, the end of the strut being cut at 90°, see Fig. 285†. We desire to know the safe load which may be put upon the strut and the additional dimensions necessary to make a satisfactory joint, the timber being Northern pine.

The surface on which the strut bears is inclined at 30° to the grain of the post. This will be the surface which will limit the strength of the strut. As Fig. 284 indicates, the safe stress on such a surface ($\theta = 30°$) is 530 lbs. per sq. in.

The area of the end of the strut will be 16 sq. ins. less the size of the tenon ($\frac{3}{4}$ in.), that is $16 - (4 \times \frac{3}{4})$ or 13 sq. ins.

Total safe load on strut $= 530 \times 13 = 6900$ lbs. (approx.).

If this be the full load (P) on the strut, then from the small force triangle *def* in Fig. 285 we see that the component of P along the post is just half this, or 3450 lbs. Now this force will tend to *shear* the end of the post along the line DC, hence we must make DC of sufficient length to prevent this, that is the area ($DC \times$ thickness

* See *Structural Details*, by Jacoby.

† For details of the construction of these joints readers may consult *Architectural Building Construction*, by Jaggard and Drury.

of post) times the safe longitudinal shearing stress (120) must be equal to 3450 lbs.,

or $$4\,(DC) \times 120 = 3450,$$

whence $DC = 7{\cdot}2$ ins., or say 8 ins., to allow for the fact that this area is cut into by the tenon.

The horizontal component fd of P is taken by the "straining head" against which the post is fitted. Since the centre line of the post and strut do not intersect on the centre line of the straining head, the post in this case will be subjected to bending as well as to tension and should be designed accordingly; see para. 219.

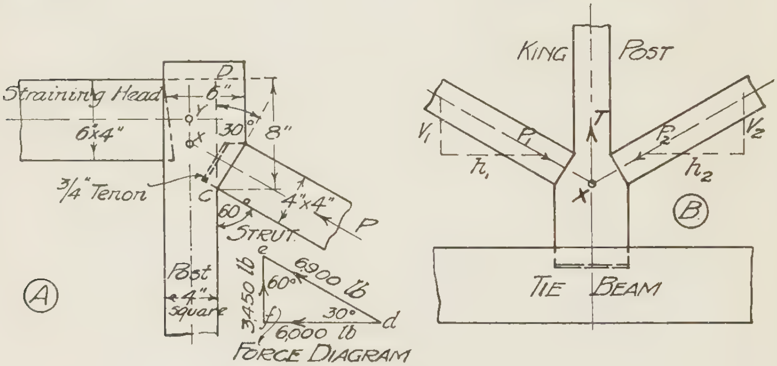


Fig. 285. Joints resisting inclined thrust.

In the case of the king post, in a king post roof truss, where two such joints are used, see Fig. 285 (B), the forces P_1 and P_2 balance each other if the loading is symmetrical. When, however, the loading is unsymmetrical—as in the case when wind pressure acts on one side of the roof—then there will be an unbalanced horizontal thrust, equal to the difference between h_1 and h_2, and some means must be adopted to take up this thrust at the joint between the post and the tie beam. The sides of the tenon may be large enough to take this thrust, or the usual metal fastenings may be constructed to function in this way. See also Fig. 288 (B).

The next type of joint to be considered in this group is that in which the depth of the main member must not be seriously reduced. A well-known example occurs between the principal rafter and the tie beam in a roof truss; see Fig. 286. A similar joint, though one not usually subjected to such large loads, occurs between the inclined struts and the principal rafter; see Fig. 288 (C). In the example illustrated in Fig. 286 we will assume that the

timber to be used is Northern pine, that the thrust acting in the rafter is 6000 lbs., and design a suitable joint for members of the given dimensions. The solution is set out in steps which may be followed in dealing with similar examples.

(a) **To find the approx. depth of the joint.** If we assume first that a notch will be cut at A with the face at right angles to the length of the tie beam, then the inclination of this face to the grain of the rafter will be 45°, and the safe stress on it (see Fig. 284) will be 675 lbs. per sq. in. This face will resist the horizontal component of P, which from the small force triangle *def* is 4240 lbs. Hence, since the face is 4 ins. wide,

$$\text{Approx. depth of notch} = \frac{4240}{675 \times 4} = 1{\cdot}56, \text{ say } 1\tfrac{5}{8} \text{ ins.}$$

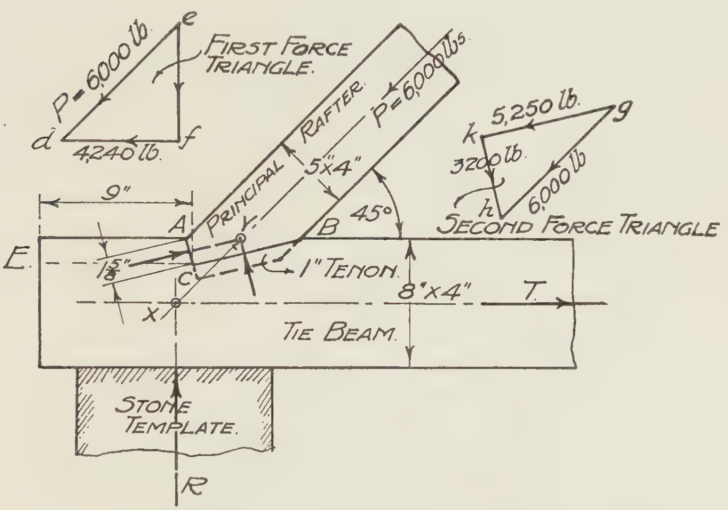


Fig. 286. Joint resisting inclined thrust.

(b) **To draw the joint and check the sizes.** The inclined surfaces AC and BC are now drawn in *at right angles to each other*, the face AC being $1\frac{5}{8}$ ins. deep; see Fig. 286. A new triangle of forces *ghk* is now drawn, with the components at right angles to the two faces AC and BC respectively. The thrust on the face AC is now seen to be 5250 lbs. This face is inclined at 60° to the grain of the rafter, for which the safe stress is 810 lbs. per sq. in. Then the required depth of face AC is

$$\frac{5250}{810 \times 4} = 1{\cdot}6 \text{ ins.},$$

so that the selected depth of $1\frac{5}{8}$ ins. may stand.

(c) **To check strength of face BC.** The face BC is inclined at 15° to the grain of the rafter, the safe stress on this surface being 390 lbs. per sq. in. The force *kh* at right angles to this surface is 3200 lbs.; hence the area required is (3200/390) or $8\frac{1}{4}$ sq. ins. Much more than this is provided even after the area occupied by the tenon has been deducted.

(d) **To check for shear along EC.** The safe longitudinal stress is 120 lbs. per sq. in. and the horizontal component of P is 4240 lbs.; see force triangle def. Hence area required is (4240/120) or 35 sq. ins. From which we have length EC is (35/4) or 9 ins. nearly.

Notes. (*a*) It is important that the axes of the rafter, tie beam and of the bearing surface should intersect, see X, Fig. 286, otherwise one of the members will be subjected to bending stresses.

(*b*) The faces AC and BC of the joint are made at right angles to each other in order that the resultant forces may intersect on the axis of the rafter; see Y, Fig. 286.

(*c*) The provision of the necessary length at EC is very important, it is in this respect that these joints as constructed in this country are usually weak.

The use of metal fastenings. No reference has yet been made to the bolts or straps which are usually used with timber joints. Bolts or straps would of course be used in the joints discussed above to hold the parts firmly together. It has been demonstrated by experiment, however, that it is rarely possible to construct a joint which depends for its strength in part upon the timber portion of the joint, and in part upon the metal work. Under test the first and deciding stage in the failure of the joint occurs either in the timber or in the metal work. It is therefore a sound principle of construction to make timber joints sufficiently strong either as timber joints alone, or as joints depending mainly upon the metal connections; see paragraphs 234–6.

234. Larger joints to resist inclined thrust. The joints discussed in the preceding paragraph are such as would occur in roof trusses up to 30 ft. span. In larger trusses, or in trusses with exceptional loads, additional means have to be adopted to produce joints of sufficient strength. Some of the methods which may be adopted are shown in Fig. 287, and others may be devised upon the principles herein laid down.

At (A) in Fig. 287 a "double step" joint is shown, in which the inclined member bears upon two surfaces or "steps" at A and B. The lowest points A and B are on different levels so that the full shear strength right to the end of the tie beam may be developed in each case. This joint must be designed so that each step takes its correct share of the thrust. The total horizontal thrust is taken over the full length AB, so that the length CA, from the end of the tie beam to the joint, need not be considerable.

At (B) and at (C) in Fig. 287 are shown joints which depend for their strength mainly upon the metal work which is used. At (B) the principal rafter abuts against a cast iron shoe, while at (C) it fits into a forged steel plate in a similar manner. In both cases the horizontal thrust along the beam is taken by a lug at B, which is housed into the beam. The depth of this lug must be proportioned

to the thrust which it has to transmit to the beam, and the length from B to C is proportioned to suit the shear strength of the beam over this length. The cast iron shoe at (B) Fig. 287 is entirely housed into the tie beam, so that, if the face at E be accurately fitted, it will assist in resisting the horizontal thrust.

The length of the under surface AB, in both the joints at (B) and (C) in Fig. 287, may be increased if necessary to reduce the

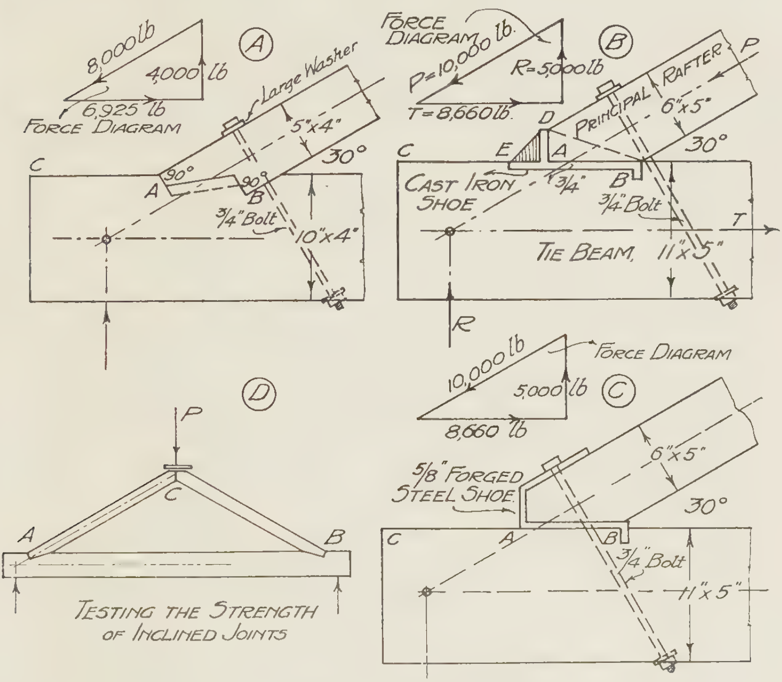


Fig. 287. Larger joints resisting inclined thrust.

bearing stress on the upper surface of the beam. The dimensions of the timber within the shoe in each case are determined in the manner already described, the principal rafter obtaining a good seating in each case. See Problems XXV, 6 to 8.

Tests of inclined joints. If facilities exist, tests on the joints described above may be carried out by making up a simple form of truss as shown at (D). Fig. 287, the joints to be tested being at A and B while the pressure from the testing machine is applied at C.

235. The use of bolts and washers in timber construction. Bolts may be used: (a) to hold together tightly the pieces of timber

forming a joint, see Fig. 290 (B), (*b*) to act in an elongated form as tension members in a truss, see Fig. 288 (B), or (*c*) to act as "pins" to resist the thrust or pull acting between two pieces of timber in a joint; see Figs. 289 (B) and 290 (A).

With regard to the first group, while we do not usually make any allowance, in calculating the strength of timber joints, for the added resistance set up by the friction between the surfaces, it is clearly an advantage to have those surfaces held together as firmly as possible, and bolts or nails can be used for this purpose.

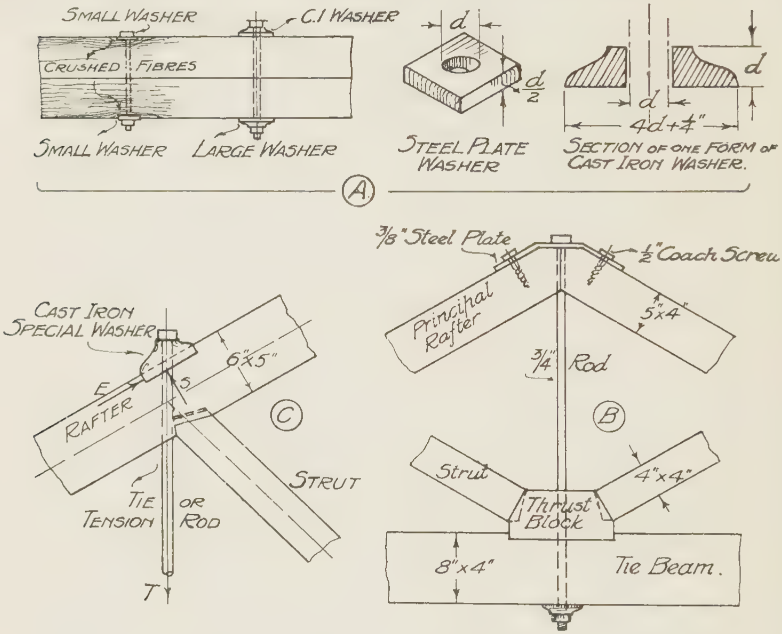


Fig. 288. Use of bolts and washers.

Washers. In this country, it is the all-too-common practice to use bolts in timber work having washers which are only slightly larger in diameter than the bolt head. The pressure which can be exerted by the bolt must thus be limited to the strength of the small area of timber on which the washer bears, and the fibres are frequently crushed in the attempt to increase the pressure. In America, it is the general practice to use much larger washers so that, if the joint requires it, the bolt may be stressed up to its full working stress. General dimensions for cast iron washers of this type are given in Fig. 288 (A). Round or square steel plate washers are also used.

Long bolts or tension bars. Fig. 288 (*B*) shows a case in which a tension rod or bolt is used to replace the king post of a roof truss (compare with Fig. 285). The resulting simplicity of the construction will be obvious. The design of the various joints can be carried out on the lines already laid down. The rod must take the total vertical thrust of the struts, hence the area of the top and bottom washers must be such as to enable them to resist the same load (about 5000 lbs. in this case). The thrust block must be sunk sufficiently to enable it to resist the unbalanced horizontal component of the thrusts in the struts; see also para. 233.

At (*C*), Fig. 288, a cast iron washer is shown which is designed to take a tension rod passing through the timber member at an angle not 90°. The thrusts E and S, from the end and bottom of the sinking respectively, must intersect on the line of action of T, the pull of the rod. The resultant of the forces E and S must equal T.

The lateral strength of bolts. Where bolts are used as "pins," and have to resist lateral pressure, their power to do so varies with the direction of the pressure in relation to the grain of the wood.

Jacoby has shown that, while the total safe pressure which can be exerted by or upon a bolt *across the grain*, see P_s in Fig. 289, equals the projected area times the safe side-grain stress, or

$$P_s = \text{projected area of bolt} \times c_s, \qquad \ldots\ldots\text{(vii)}$$

the corresponding total pressure (P) *along* the grain is only 0·6 times the projected area multiplied by the safe end-grain stress, or

$$P = 0{\cdot}6 \times \text{projected area of bolt} \times c. \qquad \ldots\ldots\text{(viii)}$$

The considerable reduction in the total pressure (P) which can be transmitted through a bolt pressing along the grain is due to the fact that, round the half-circle of contact between the bolt and the timber, the pressure from the bolt and the lie of the grain are changing their relative directions. We cannot therefore count on subjecting the whole of the "projected area" to the full end-grain stress (c), as is possible in the case of a homogeneous material like steel.

(Strictly the constant 0·6 only applies when the stresses c and c_s are in the ratio of 4 : 1, which is, however, approximately correct for most of the common structural timbers.)

At intermediate positions between the positions of the forces P and P_s, the magnitude of the force P is affected by the varying strength of the timber; see Fig. 284. Both Jacoby (see *Structural Details*) and Gardner, in the paper already referred to, have analysed this problem very thoroughly and readers should consult these authorities for further details.

As is shown by the writers just mentioned, the problem is still further complicated by the fact that, unless the bolts are of very stout section, they will bend considerably under load and thus alter the distribution of stress along their length; see Fig. 289 (B), which

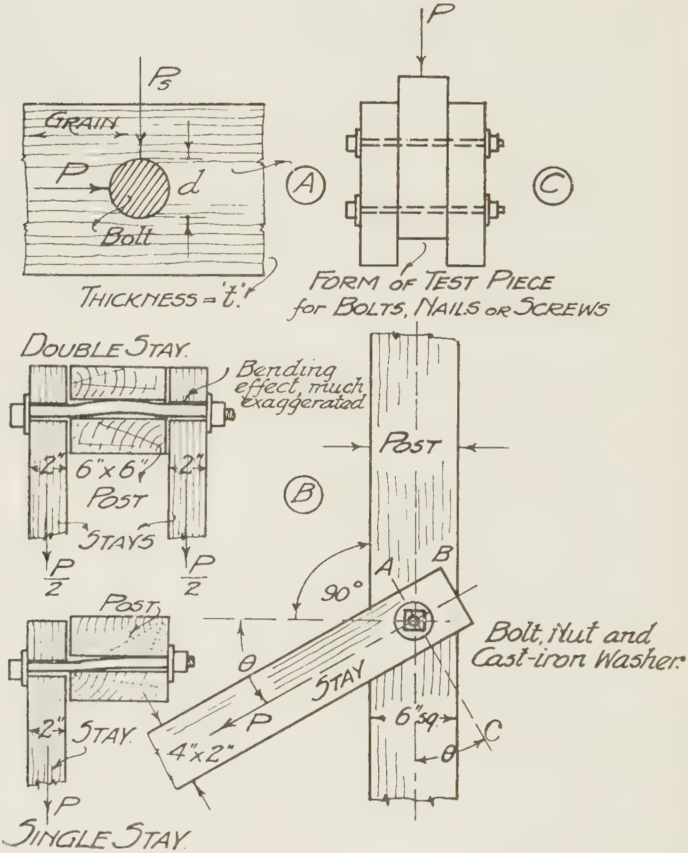


Fig. 289. The lateral strength of bolts.

also shows an inclined stay secured by a bolt to a post or main member of stouter section. For accurate design the strength of the bolts in bending, and the effect of this bending upon the distribution of the bearing stress on the timber, should be considered. It will thus be seen that the design of these apparently simple joints involves unexpected difficulties. As furnishing a method which is probably sufficiently accurate for most practical

cases of inclined stays, the authors suggest the use of the following approximate rule:—

Approximate rule for bolted stays. To find P, the maximum pull per bolt which a stay will carry, let θ be the angle between the stay and a line drawn at right angles to the main member, see Fig. 289 (B). Then:

(a) When θ is less than 45°, take the minimum allowable bearing stress on the bolt to be equal to c_θ, and use the expression (vii) above, modified as follows:

$$P = \textbf{projected area of bolt in stay} \times c_\theta. \qquad \text{......(ix)}$$

(b) When θ exceeds 45°, take the minimum pull to be equal to P *along the grain*, as in the expression (viii) or

$$P = 0{\cdot}6 \times \textbf{projected area of bolt in stay} \times c. \qquad \text{......(x)}$$

The stresses c_θ and c can be obtained from Fig. 284; see Example 1 below.

It should be observed that, in securing tension members by bolts, it is generally most economical to use few bolts of large section, the width of the tension member being usually wide enough to accommodate them without difficulty; see Example 2 below.

Where the bolts may be subjected to severe bending, large washers should not as a rule be used, a large amount of initial tensile stress cannot then be accidentally put upon the bolts, which would reduce their power to resist the bending stresses.

Experiment. Experiments upon the lateral strength of bolts (and also of screws and nails) may be carried out by means of a test piece built up as shown in Fig. 289 (C).

Example 1. *Find the safe load which may be put upon the double stay in Fig.* 289 (B), *if the material is white deal, the angle θ is* 40°, *and a single $\frac{3}{4}$-in. bolt is used.*

The inclination of the pull being just 40°, the safe stress, from Fig. 284, is 525 lbs. Then

$$\text{Total pull} = 2\,(525 \times 2 \times \tfrac{3}{4}) = 1575 \text{ lbs.}$$

The length AB beyond the bolt should be such as to prevent failure by shear. Two surfaces would have to be sheared and the safe shearing stress is 100 lbs. Then (for one stay)

$$\text{Minimum length} = \frac{787{\cdot}5}{100 \times 2 \times 2} = 2 \text{ ins. nearly.}$$

A greater length than this would be allowed, since there is always a tendency for these ends to split, owing to the alternate wetting and drying to which they are subjected. *A safe rule is to make this distance—and also the distance between bolts in a row—equal to six times the diameter of the bolt.* If several bolts are used in a joint they may each be assumed to take their full load.

Example 2. *The tie beam (9 ins. × 4 ins.) in a large truss carries a tensile load of 15,000 lbs. Design a bolted connection for a joint in this member using timber fish plates. The timber is Northern pine.*

Fig. 290 (A) will sufficiently explain the form which the connection is to take. The fish plates have each been made more than half the thickness of the main member, and we may thus assume that there is no need to check their strength separately. (The joint should properly be checked for every possible failure; see Riveted Joints, Chap. XVIII.)

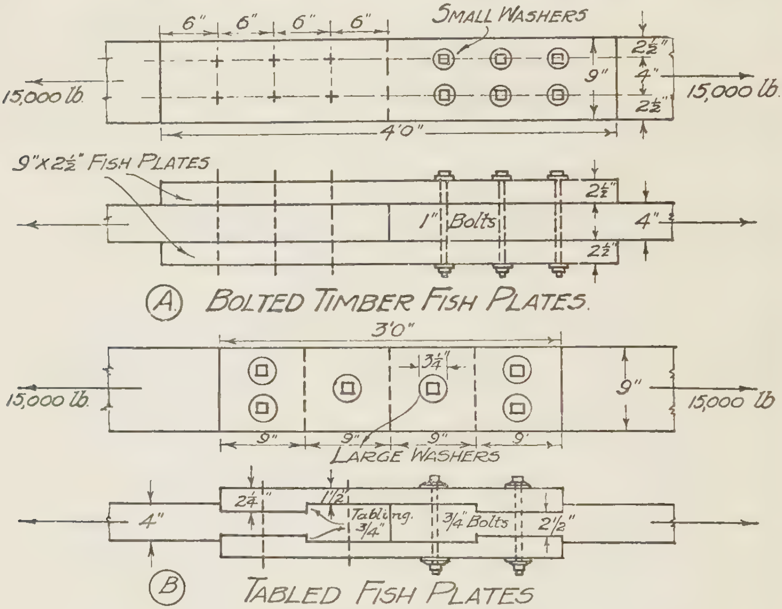


Fig. 290. Tension joints in timber.

An *even* number of bolts should preferably be used, say 6 on each side. Then

$$\text{Load per bolt} = \frac{15{,}000}{6} = 2500 \text{ lbs.},$$

$$\text{Diameter of each bolt} = \frac{2500}{0{\cdot}6 \times 1100 \times 4} = 0{\cdot}95, \text{ say 1 in.}$$

If the bolts are spaced at $(6\,d)$ or 6 ins. apart and at the same distance from the ends of the timbers, then the joint will be safe against shear and the arrangement will be as shown in Fig. 290 (A). As already explained, small washers are preferable in such a case under normal conditions.

236. Tabled joints. Fig. 290 (B) shows a somewhat more compact form of tension joint than that shown in Fig. 290 (A). This form of joint depends for its strength upon the "tables" formed

in the timber. The same result might have been achieved by the use of hardwood keys in the place of the tables. Tabling can be used in a variety of ways and makes a relatively cheap joint since no special metal parts are necessary. The design of such a joint is set out below. Large washers should be used in this case, since the main purpose of the bolts is to hold the timbers firmly together.

Example. *The joint is to be designed for the same material and load as in Example 2, para.* 235.

To carry a load of 15,000 lbs. in tension a net area of timber of (15,000/1100) or 13·6 sq. ins. is required, so that neither the minimum sectional area of the main member nor the combined minimum area of the two fish plates should be less than this.

Since the end-grain strength of the timber is also 1100 lbs. per sq. in., a minimum total "bearing" area is required on the effective ends of the tables (and notches) of 13·6 sq. ins.

As there is a table on each side of the joint, the combined width being 18 ins., the necessary depth = 13·6/18 = $\frac{3}{4}$ in. nearly. The main member is thus reduced to a thickness of $2\frac{1}{2}$ ins. at the sinkings, the net area in tension is thus ($9 \times 2\frac{1}{2}$) or 22·5 sq. ins., which is ample.

The safe longitudinal shear stress being 120 lbs. per sq. in., then the total shearing area required = 15,000/120 = 125 sq. ins.

This is divided between two tables each 9 ins. wide, the length of each table is therefore 125/18 = 7 ins. This has been increased to 9 ins. to provide allowance for the bolt holes.

The plates may be given a minimum thickness of $1\frac{1}{2}$ ins. each, the total thickness being then $2\frac{1}{4}$ ins. Two $\frac{3}{4}$-in. bolts with large washers should be used at each table, with single bolts in the other two spacings; see Fig. 290 (B).

237. The lateral strength of screws and nails. Both the "holding" and the lateral strength of screws and nails have received considerable attention from experimenters in America. The results are usually based upon experiment rather than upon any theoretical analysis, so that it is not possible to give a simple rule which can be used under all circumstances.

Lateral strength of single wire nails. The following values may be used:

6 in. nails—200 lbs.,

5 in. nails—150 lbs.,

4 in. nails—100 lbs.

The nails should be driven at least two-thirds of their length into the main member. The same values may be used for cut nails.

Where **screws** are used they may be treated on the same basis as bolts provided that at least two-thirds of their length penetrates the main timber.

Timber Beams

238. The design of timber beams. Since practically all timber beams are of rectangular section, their design should not give rise to any serious difficulties if the work done in Chaps. XIII, XIV and XV, on the strength of beams, has been properly understood. In calculating the strength of a timber beam the stresses given in Table XV for tension, compression and shear (in bending) should be used. Having obtained a suitable mid-section this should be checked for maximum shear stress ($s_{\text{max.}} = \frac{3}{2}\, S/bd$, see para. 137).

The deflection of timber beams should be limited to $\frac{1}{400}$ of the span. In calculating this deflection, however, it is necessary to remember that time is an important factor in the straining of a timber beam, the deflection for a long-continued load being greater than it would be if the load be applied only for a short time—as in a test. *If a large proportion of the load to be carried by a timber beam is "dead load", that is a loading which does not vary over long periods of time, then it is usual to multiply this part of the load by a factor, which may vary from* 1·5 *to* 2, *in calculating the maximum deflection.*

Large beams should generally be designed to give not more than a specified deflection, while smaller beams may be designed on a strength basis. While deflection may be kept low by using deep beams, it must be remembered that shear stresses may thereby be increased. Where the dimensions of the beams are already fixed—for example by the market sizes available—the calculations should aim at finding the most economical spacing for the beams; see Example 2, para. 133.

239. Connections to timber beams. In the construction of floors it is the common practice in this country to "frame" the beams and joists together by means of notches or mortises and tenons (see any book on Building Construction), and it is frequently stated that some of the joints there set out do not seriously weaken the beams. Unfortunately this is far from true, as the reader may himself demonstrate by carrying out a series of simple tests on specimen beams such as are set out in Fig. 291. Other test pieces may be devised, but as the results would show, practically all joints which require cuts to be made in a beam will reduce its strength to a surprising extent.

This weakening effect appears to be due not so much to the reduction of the section of the beam, as to the sudden changes in section which are thereby occasioned. Thus a beam with a gradual change in section as at (*c*) in Fig. 291 will be stronger than the notched beam at (*b*), though the reduction in depth at the ends is the same in each case. In the case of beam (*b*) failure will generally

take place by the beam splitting at the corners A and B, indicating a concentration of stress at these points. It follows that any form of floor construction which avoids unnecessary cutting—such as that in which metal joist hangers are used—is to be commended.

240. Elastic limit. Modulus of rupture. If the timber beam shown in Fig. 292 be loaded with gradually increasing loads, then the deflection produced will be proportional to the loading at each stage as in other elastic materials. This will continue until the

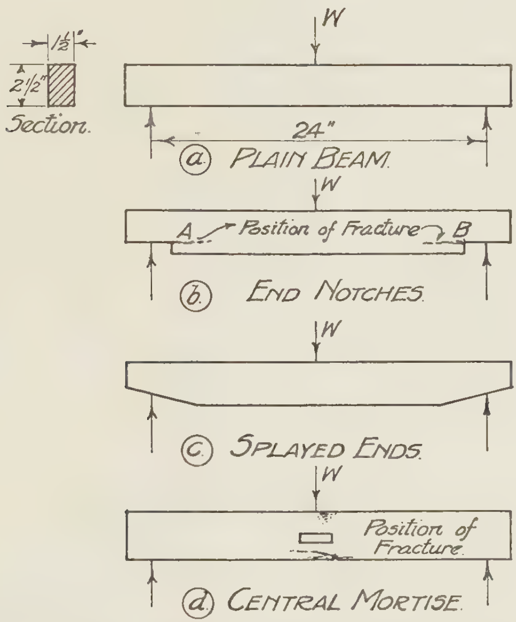


Fig. 291. Effect of cutting on strength of beams (experiment).

Elastic Limit (or more correctly, the Limit of Proportionality) is reached, see the straight portion MN of the graph in Fig. 292. Similar results will be obtained with central loading.

It may be shown that the stress due to bending when the elastic limit is reached is usually close to the elastic limit stress of the timber in direct compression. Bending tests may therefore be used to find the *compressive elastic limit stress* of the timber, in preference to more direct but more elaborate methods involving the measurement of strain on a compression test piece.

The *tensile elastic limit* is usually very near to the ultimate tensile stress. It cannot be ascertained by a bending test as in the

case of compression, but must be found from direct tension tests. Since beams are usually designed on the basis of the working compressive stress, see Table XV, the elastic limit stress in tension is not of great practical importance and further laboratory details need not be given here.

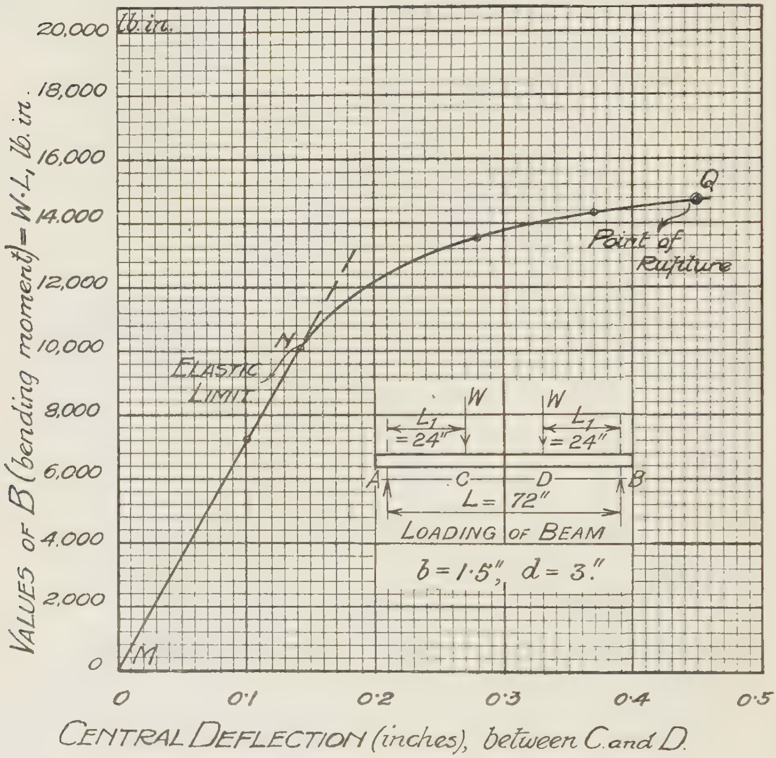


Fig. 292. Test to destruction of a timber beam in white deal.

Modulus of rupture. If the loading of the beam shown in Fig. 292 be carried beyond the elastic limit at N, the deflection beyond N will increase more rapidly than the loading, and the graph will become curved as shown; this will continue up to the point of rupture at Q. Such a beam will usually show the steady failure of the upper fibres of the beam *by compression*, from the point N right up to the point Q, the crumpling of the fibres extending deeper and deeper into the beam. Finally the beam will fracture suddenly by the tensile failure of the fibres on the underside of the beam. (That the final failure is by tension will be shown by the fact that,

should any defects be present on the underside of the beam, such as knots or cross-grain, then the beam will fail at a figure which is lower than the average.)

Taking the average of a number of tests of beams *from the same timber*, it will be found that the breaking load (B.W.) bears a direct relation to the dimensions of the beam. This fact is used as the basis of a method for comparing the strengths of various timbers. The following statement will explain how the comparison is made.

Although the theory of elastic bending does not apply beyond the elastic limit of the material (see para. 116), it has been found that, at the time of rupture of a rectangular beam, the relation between the breaking load and the dimensions of the beam are approximately similar to those in simple bending and may be expressed in the form "$B = fZ$". In this case, however, a constant f_r, the **Modulus of Rupture**, is substituted for "f", and B is the bending moment at rupture and Z the modulus of the section; or

$$\text{Bending moment at rupture} = f_r Z = (\text{modulus of rupture}) \times Z \quad \ldots\ldots\text{(xi)}$$

The method is most commonly used in simple breaking tests on centrally loaded rectangular beams; in this case we have $B = (\text{B.W.}) \times L/4$, and $Z = bd^2/6$, whence

$$\frac{(\text{B.W.})\,L}{4} = f_r \frac{bd^2}{6},$$

or

$$f_r = \text{Modulus of rupture} = \frac{3}{2}\,\frac{(\text{B.W.})\,L}{bd^2}. \quad \ldots\ldots\text{(xii)}$$

The values of f_r obtained in this way are approximately constant for the same kind of timber and may therefore be used as a means for comparing the strengths of various timbers. Average values of f_r for the timbers dealt with in this chapter are: (I) Pitch Pine 10,000 lbs. per sq. in.; (II) Oak 9000 lbs. per sq. in.; (III) Northern Pine 7500 lbs. per sq. in.; (IV) White Deal 6500 lbs. per sq. in.

If the expression (xi) is divided by a factor of safety (F), the value of which in timber may range between 5 and 8, we obtain an expression for the safe bending moment which may be put upon the timber in terms of f_r, thus

$$\text{Safe bending moment} = \frac{\text{bending moment at rupture}}{\text{factor of safety}} = \frac{f_r Z}{F}. \quad \ldots\ldots\text{(xiii)}$$

(**Note.** This expression is sometimes used to form the basis of a method of designing timber beams, and in many text-books the value of the bending moment for various forms of beam fixing and loading is inserted, and a long series of expressions built up, in a manner similar to that

explained in connection with expression (xii) above. In this form the method has little to commend it; it is not so simple as it appears to be and it does not give any indication of the stiffness of the beam (see para. 142), nor of the magnitude of the longitudinal shear stresses, which we have shown to be so important (see para. 225). There can be little doubt that the method of design outlined at the commencement of para. 238, and which is properly based upon the theory of bending, is the most suitable one to use for beams in timber as well as in other materials. This method should always be used in preference to that contained in expression (xiii), which is merely mentioned here in order to make this statement concerning timber beams as complete as possible. The modulus of rupture figure (f_r) is, however, of considerable value in comparing the results obtained from tests on timber beams.)

Problems XXV

1. A tension test piece is to be cut from pitch pine, the outside dimensions being 3 ins. by $1\frac{1}{2}$ ins.; see Fig. 280. It is expected that the ultimate stress in tension may reach 18,000 lbs. per sq. in. Using a longitudinal shear stress of 200 lbs. per sq. in. and an end-grain compressive stress of 2000 lbs. per sq. in., design a test piece similar to that shown in Fig. 280. Give the *minimum dimensions* of (a) the shoulders at E, (b) the distance EF, and (c) the distance DF.

2. In a carefully constructed timber truss a strut 4 ins. by 4 ins. in section and 8 ft. long is used. If the timber be Northern pine and the ends can be assumed to be "flat ends", calculate the total safe compressive load which may be put upon it, using the Johnson-Euler column formula.

3. Using a straight-line formula, determine the safe load which may be put upon "flat-ended" struts in 4 in. by 3 in. spruce as used for supporting concrete shuttering. The maximum unsupported length of the struts is 8 ft.

4. If the pitch pine shore mentioned in the Example, para. 227, be made up of two 9 in. by $4\frac{1}{2}$ in. pieces bolted together, calculate the safe load which may be put upon it, if the shore, by reason of the method of construction, can only be looked upon as two *separate* columns, each bearing half the total load. The ends may be assumed to be "flat ends" in this case.

5. See Fig. 285. As sometimes occurs the centre lines of the inclined strut, the vertical post and the straining head do not intersect in the same point. If the two points X and Y, in which the intersections take place, are $1\frac{1}{2}$ ins. apart, calculate (a) the maximum bending moment produced, and (b) the maximum stress induced in the post at section C. (The maximum stress at C will be made up of the direct stress and the bending stress. The same value of the bending moment should be taken as at X. The direct load on the post is 3450 lbs.)

6. Calculate the theoretical depth of the inclined face AC in the joint shown in Fig. 286, and also the minimum length of EC, if the inclined force in the principal rafter be of the same magnitude (6000 lbs.) but acts at an angle of 30°.

7. Calculate the net total depth of the inclined faces at A and B for the forces indicated in Fig. 287 (A), if the truss be in Northern pine.

If the horizontal distance between A and B be approximately 6 ins., what should the distance AC be made?

8. For the forces indicated in Fig. 287 (B), calculate (*a*) the depth of the vertical face at DA, (*b*) the minimum length of BC, (*c*) the depth of the lug at B to take the balance of the horizontal thrust, if the cast iron shoe be let $\frac{3}{4}$ in. into the tie beam at E, and (*d*) the thickness of the lug at B, if the maximum bending stress on the cast iron is not to exceed 3500 lbs. per sq. in. The truss is in Northern pine.

9. Ignoring the strengthening effect of the bolt in Fig. 287 (C), calculate (*a*) the depth of the lug at B, (*b*) the minimum distance from B to the end C of the tie beam, and (*c*) the thickness of the lug, if the bending stress induced must not exceed 16,000 lbs. per sq. in.

10. If the bolt passing through the inclined member at (C) in Fig. 288 is $\frac{3}{4}$ in. in diameter, the inclination of the rafter being 30°, find the minimum dimensions of the side and base of the C.I. washer to resist the forces E and S, when the bolt is fully stressed to 16,000 lbs. per sq. in. The truss is 5 ins. thick and in Northern pine.

11. If the inclined stays mentioned in Prob. 7, Chap. XI, for a derrick tower, be inclined to the horizontal at 35° and be made from 6 in. by $2\frac{1}{2}$ in. white deal, calculate the size and number of bolts to be used, (*a*) to splice each member, and (*b*) to connect the members to the uprights, if the maximum forces to be carried are respectively 1200, 2400 and 3600 lbs.

12. A beam of white deal is required to carry a uniformly distributed load of 2 tons over a span of 12 ft. Using the working stresses given in Table XV, calculate suitable dimensions for the beam, making the depth $\frac{1}{15}$ the span. What is the ratio between the deflection and the span at full load? What is the maximum shear stress?

13. From the values given in Fig. 292, calculate (*a*) the elastic limit stress, (*b*) the value of E, the modulus of elasticity, and (*c*) the value of f_r, the modulus of rupture.

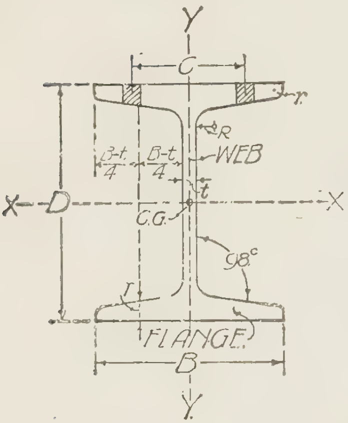
14. Using the value for the modulus of rupture found in Prob. 13, calculate the breaking weight (B.W.) for a uniformly distributed load on the beam in Prob. 12. From these figures, find the value of the factor of safety (F) in this case.

APPENDIX I

BRITISH STANDARD SECTIONS[1]

(a) OLD BRITISH STANDARD BEAMS

(original sections)



DIMENSIONS AND PROPERTIES OF OLD **I** BEAMS IN INCH UNITS

Angle between web and flange = 98

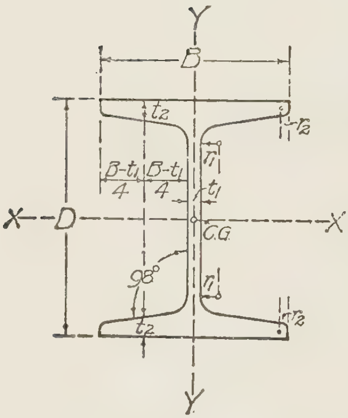
Reference Mark	Size in Inches D × B	Weight per Foot in lbs.	Area in Square Inches	Diagram: Web t	Diagram: Flange T	Diagram: Radius R	Diagram: Radius r	Moments of Inertia Inch⁴ units: Maximum About X–X	Moments of Inertia Inch⁴ units: Minimum About Y–Y	Radii of Gyration Inches: Max. About X–X	Radii of Gyration Inches: Min. About Y–Y	Modulus of Section Max. About X–X Inch³ units	Centres of Holes C Inches	Reference Mark
	1	2	3	4	5	6	7	8	9	10	11	12	13	
B.S.B. 30	24 × 7½	100	29·4	·6	1·07	·7	·35	2654	66·92	9·5	1·5	221·1	4·5	B.S.B. 30
,, 29	20 × 7½	89	26·17	·6	1·01	·7	·35	1670	62·63	7·99	1·54	167·0	4·5	,, 29
,, 28	18 × 7	75	22·06	·55	·928	·65	·325	1149	47·04	7·21	1·46	127·6	4·0	,, 28
,, 27	16 × 6	62	18·23	·55	·847	·65	·325	725·7	27·08	6·31	1·21	90·71	3·5	,, 27
,, 26	15 × 6	59	17·35	·5	·88	·6	·3	628·9	28·22	6·02	1·27	83·85	3·5	,, 26
,, 25	15 × 5	42	12·35	·42	·647	·52	·26	428·0	11·81	5·88	·978	57·06	2·75	,, 25
,, 24	14 × 6	57	16·76	·5	·873	·6	·3	532·9	27·96	5·63	1·29	76·12	3·5	,, 24
,, 23	14 × 6	46	13·53	·4	·698	·5	·25	440·5	21·6	5·7	1·26	62·92	3·5	,, 23
,, 22	12 × 6	54	15·88	·5	·883	·6	·3	375·5	28·3	4·86	1·33	62·58	3·5	,, 22
,, 21	12 × 6	44	12·94	·4	·717	·5	·25	315·3	22·27	4·93	1·31	52·55	3·5	,, 21

[1] Extracted by permission of the British Engineering Standards Association from its Report, No. 6, 1924, *Dimensions and Properties of British Standard Rolled Steel Sections for Structural Purposes.* Official copies may be obtained from the Secretary, 28 Victoria Street, London, S.W. 1, price 5*s.* 4*d.*, post free.

	1	2	3	4	5	6	7	8	9	10	11	12	13	
B.S.B. 20	12 × 5	32	9·41	·35	·55	·45	·225	220	9·753	4·83	1·01	36·66	2·75	B.S.B. 20
,, 19	10 × 8	70	20·6	·6	·97	·7	·35	344·9	71·67	4·09	1·86	68·98	4·75	,, 19
,, 18	10 × 6	42	12·35	·4	·736	·5	·25	211·5	22·95	4·13	1·36	42·3	3·5	,, 18
,, 17	10 × 5	30	8·82	·36	·552	·46	·23	145·6	9·79	4·06	1·05	29·12	2·75	,, 17
,, 16	9 × 7	58	17·06	·55	·924	·65	·325	229·5	46·3	3·66	1·64	51·0	4·0	,, 16
,, 15	9 × 4	21	6·176	·3	·46	·4	·2	81·1	4·2	3·62	·824	18·02	2·25	,, 15
,, 14	8 × 6	35	10·29	·44	·597	·54	·27	110·5	17·95	3·27	1·32	27·62	3·5	,, 14
,, 13	8 × 5	28	8·24	·35	·575	·45	·225	89·32	10·26	3·29	1·11	22·33	2·75	,, 13
,, 12	8 × 4	18	5·294	·28	·402	·38	·19	55·69	3·578	3·24	·822	13·92	2·25	,, 12
,, 11	7 × 4	16	4·706	·25	·387	·35	·175	39·21	3·414	2·88	·851	11·2	2·25	,, 11
,, 10	6 × 5	25	7·35	·41	·52	·51	·255	43·61	9·116	2·43	1·11	14·53	2·75	,, 10
,, 9	6 × 4½	20	5·88	·37	·431	·47	·235	34·62	5·415	2·42	·959	11·54	2·5	,, 9
,, 8	6 × 3	12	3·53	·26	·348	·36	·18	20·21	1·339	2·39	·616	6·736	1·5	,, 8
,, 7	5 × 4½	18	5·29	·29	·448	·39	·195	22·69	5·664	2·07	1·03	9·076	2·5	,, 7
,, 6	5 × 3	11	3·235	·22	·376	·32	·16	13·61	1·462	2·05	·672	5·444	1·5	,, 6
,, 5	4¾ × 1¾	6·5	1·912	·18	·325	·28	·14	6·73	·263	1·87	·37	2·833	—	,, 5
,, 4	4 × 3	9·5	2·794	·22	·336	·32	·16	7·52	1·281	1·64	·677	3·76	1·5	,, 4
,, 3	4 × 1¾	5	1·47	·17	·24	·27	·135	3·668	·186	1·58	·355	1·834	—	,, 3
,, 2	3 × 3	8·5	2·5	·2	·332	·3	·15	3·787	1·262	1·23	·71	2·524	1·5	,, 2
,, 1	3 × 1½	4	1·176	·16	·248	·26	·13	1·659	·124	1·18	·324	1·106	—	,, 1

APPENDIX I (*cont.*)

(*b*) NEW BRITISH STANDARD BEAMS



DIMENSIONS AND PROPERTIES OF NEW I BEAMS IN INCH UNITS

Angle between web and flange = 98°

Reference Mark	Size in Inches D × B	Wt. per Foot in lbs.	Sectional Area in Square Inches	Standard thickness: Web t_1	Standard thickness: Flange t_2	Radii: Root r_1	Radii: Toe r_2	Moments of Inertia Inch⁴ units: Maximum About X–X	Moments of Inertia Inch⁴ units: Minimum About Y–Y	Radii of Gyration Inches: Max. About X–X	Radii of Gyration Inches: Min. About Y–Y	Moduli of Section Inch³ units: Max. About X–X	Moduli of Section Inch³ units: Min. About Y–Y	Centres of Holes C Inches	Reference Mark
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	
	"A." Girder Sections.														
N.B.S.B. 1	3 × 1½	4	1·177	·16	·249	·25	·12	1·66	·125	1·18	·32	1·10	·167	¾	N.B.S.B. 1
" 2	4 × 1¾	5	1·470	·17	·239	·27	·13	3·66	·186	1·57	·35	1·83	·213	⅞	" 2
" 3	4½ × 2	7	2·060	·19	·322	·29	·14	6·65	·383	1·79	·43	2·95	·383	1	" 3
" 4	5 × 2½	9	2·647	·20	·347	·33	·16	10·91	·789	2·03	·54	4·36	·631	1¼	" 4
" 5	6 × 3	12	3·533	·23	·377	·37	·18	20·98	1·46	2·43	·64	6·99	·974	1½	" 5
" 6	7 × 3½	15	4·416	·25	·398	·41	·20	35·90	2·40	2·85	·73	10·25	1·37	1¾	" 6

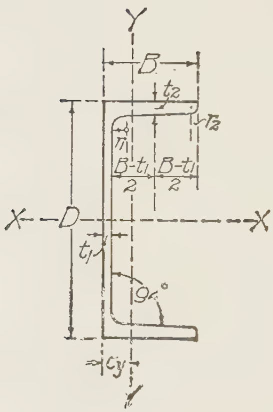
		1	2	3	4	5	6	7	8	9	10	11	12	13	14		
N.B.S.B.	7	8 × 4	18	5·296	·28	·398	·45	·22	55·62	3·50	3·24	·81	13·90	1·75	$2\frac{1}{4}$	N.B.S.B.	7
,,	8	9 × 4	21	6·177	·30	·457	·45	·22	81·12	4·14	3·62	·82	18·02	2·07	$2\frac{1}{4}$	,,	8
,,	9	10 × $4\frac{1}{2}$	25	7·354	·30	·505	·49	·24	122·3	6·48	4·07	·93	24·46	2·88	$2\frac{1}{2}$	,,	9
,,	10	12 × 5	30	8·827	·33	·507	·53	·26	206·9	8·77	4·84	·99	34·48	3·50	$2\frac{3}{4}$	,,	10
,,	11	13 × 5	35	10·29	·35	·604	·53	·26	283·5	10·81	5·24	1·02	43·61	4·32	$2\frac{3}{4}$	,,	11
,,	12	14 × $5\frac{1}{2}$	40	11·76	·37	·627	·57	·28	377·0	14·78	5·66	1·12	53·86	5·37	3	,,	12
,,	13	15 × 6	45	13·23	·38	·655	·61	·30	491·9	19·87	6·09	1·22	65·58	6·62	$3\frac{1}{2}$	,,	13
,,	14	16 × 6	50	14·70	·40	·726	·61	·30	618·1	22·46	6·48	1·23	77·26	7·48	$3\frac{1}{2}$	,,	14
,,	15	18 × 6	55	16·18	·42	·757	·61	·30	841·7	23·63	7·21	1·20	93·52	7·87	$3\frac{1}{2}$	,,	15
,,	16	20 × $6\frac{1}{2}$	65	19·11	·45	·820	·65	·32	1226	32·55	8·00	1·30	122·6	10·01	$3\frac{3}{4}$	,,	16
,,	17	22 × 7	75	22·06	·50	·834	·69	·34	1676	41·06	8·71	1·36	152·4	11·73	4	,,	17
,,	18	24 × $7\frac{1}{2}$	90	26·46	·52	·984	·73	·36	2443	60·43	9·60	1·51	203·6	16·11	$4\frac{1}{2}$	,,	18

"B." Heavy Beams and Pillars.

N.B.S.H.B.	1	4 × 3	10	2·94	·24	·347	·37	·18	7·78	1·32	1·62	·67	3·89	·88	$1\frac{1}{2}$	N.B.S.H.B.	1
,,	2	5 × $4\frac{1}{2}$	20	5·88	·29	·513	·49	·24	25	6·59	2·06	1·05	10·01	2·92	$2\frac{1}{2}$	,,	2
,,	3	6 × 5	25	7·35	·33	·561	·53	·26	45·1	9·87	2·47	1·15	15·05	3·95	$2\frac{3}{4}$	,,	3
,,	4	8 × 6	35	10·29	·35	·648	·61	·30	115	19·54	3·34	1·37	28·76	6·51	$3\frac{1}{2}$	,,	4
,,	5	9 × 7	50	14·71	·40	·825	·69	·34	208	40·16	3·76	1·65	46·25	11·47	4	,,	5
,,	6	10 × 6	40	11·77	·36	·709	·61	·30	204	21·75	4·17	1·36	40·96	7·25	$3\frac{1}{2}$	,,	6
,,	7	10 × 8	55	16·17	·40	·783	·77	·38	288	54·74	4·22	1·84	57·73	13·68	$4\frac{3}{4}$	,,	7
,,	8	12 × 8	65	19·12	·43	·904	·77	·38	487	65·18	5·05	1·84	81·29	16·29	,,	,,	8
,,	9	14 × 8	70	20·58	·46	·920	·77	·38	705	66·67	5·85	1·80	100·7	16·66	,,	,,	9
,,	10	16 × 8	75	22·06	·48	·938	·77	·38	973	68·30	6·64	1·75	121·7	17·07	,,	,,	10
,,	11	18 × 8	80	23·52	·50	·950	·77	·38	1292	69·42	7·41	1·71	143·5	17·35	,,	,,	11

APPENDIX I (*cont.*)

(*c*) NEW BRITISH STANDARD CHANNELS



DIMENSIONS AND PROPERTIES OF NEW STANDARD CHANNELS IN INCH UNITS

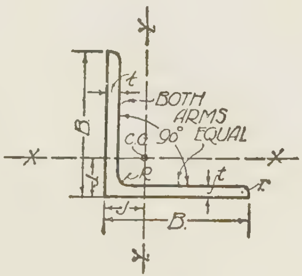
Angle between web and flange = 92°

Reference Mark	Size in Inches D × B	Weight per Foot in lbs.	Sectional Area in Square Inches	Standard thickness		Radii		Moments of Inertia Inch⁴ units		Radii of Gyration Inches		Moduli of Section Inch³ units		Centre of Gravity from Back (Cy) Inches	Reference Mark
				Web t_1	Flange t_2	Root r_1	Toe r_2	Maximum About X–X	Minimum About Y–Y	Max. About X–X	Min. About Y–Y	Max. About X–X	Min. About Y–Y		
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	
N.B.S.C. 1	3 × 1½	4·60	1·35	·20	·28	·30	·15	1·82	·261	1·16	·439	1·21	·255	·476	N.B.S.C. 1
" 2	4 × 2	7·09	2·08	·24	·31	·36	·18	5·06	·703	1·55	·581	2·53	·502	·599	" 2
" 3	5 × 2½	10·22	3·00	·25	·38	·42	·21	11·87	1·64	1·98	·739	4·74	·950	·773	" 3
" 4	6 × 3	12·41	3·65	·25	·38	·48	·24	21·27	2·82	2·41	·880	7·09	1·33	·890	" 4
" 5	6 × 3½	16·48	4·84	·28	·48	·54	·27	28·88	5·29	2·44	1·04	9·62	2·24	1·14	" 5
" 6	7 × 3	14·22	4·18	·26	·42	·48	·24	32·75	3·25	2·79	·882	9·35	1·53	·875	" 6

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	
N.B.S.C. 7	7 × 3½	18·28	5·37	·30	·50	·54	·27	42·83	5·83	2·82	1·04	12·23	2·42	1·09	N.B.S.C. 7
,, 8	8 × 3	15·96	4·69	·28	·44	·48	·24	46·72	3·57	3·15	·873	11·68	1·65	·834	,, 8
,, 9	8 × 3½	20·21	5·94	·32	·52	·54	·27	60·57	6·37	3·19	1·03	15·14	2·59	1·04	,, 9
,, 10	9 × 3	17·46	5·13	·30	·44	·48	·24	62·52	3·75	3·48	·855	13·89	1·69	·781	,, 10
,, 11	9 × 3½	22·27	6·54	·34	·54	·54	·27	82·61	6·89	3·55	1·02	18·35	2·76	1·00	,, 11
,, 12	10 × 3	19·28	5·67	·32	·45	·48	·24	82·66	3·98	3·81	·838	16·53	1·76	·742	,, 12
,, 13	10 × 3½	24·46	7·19	·36	·56	·54	·27	109·5	7·42	3·90	1·01	21·90	2·92	·965	,, 13
,, 14	12 × 3½	25·25	7·42	·35	·50	·54	·27	156·3	7·06	4·58	·975	26·06	2·66	·849	,, 14
,, 15	,,	29·23	8·59	·40	·60	·54	·27	180·2	8·43	4·58	·991	30·04	3·24	·901	,, 15
,, 16	12 × 4	31·33	9·21	·40	·60	·60	·30	200·0	12·12	4·66	1·14	33·34	4·11	1·05	,, 16
,, 17	15 × 4	36·37	10·69	·41	·62	·60	·30	349·0	13·33	5·71	1·11	46·54	4·39	·967	,, 17
,, 18	17 × 4	44·34	13·04	·48	·68	·60	·30	520·1	15·26	6·31	1·08	61·19	4·95	·920	,, 18

APPENDIX I (*cont.*)

(*d*) BRITISH STANDARD EQUAL ANGLES



DIMENSIONS AND PROPERTIES OF EQUAL ANGLES IN INCH UNITS

Reference Mark	Size and Thickness B × B × t (Inches)	Weight per Foot in lbs.	Area in Square Inches	Radii: Root R	Radii: Toe r	Dimension J	Moment of Inertia XX	Modulus of Section XX	Least Radius of Gyration
	1	2	3	4	5	6	7	8	9
B.S.E.A. 16	8 × 8 × $\frac{1}{2}$	26·35	7·75	·600	·425	2·15	47·4	8·10	1·58
,, 16	,, ,, $\frac{5}{8}$	32·67	9·609	·600	·425	2·20	58·2	10·03	1·57
,, 16	,, ,, $\frac{3}{4}$	38·89	11·437	·600	·425	2·25	68·5	11·91	1·56
,, 16	,, ,, $\frac{7}{8}$	45·00	13·234	·600	·425	2·30	78·41	13·76	1·56
,, 14	6 × 6 × $\frac{3}{8}$	14·83	4·362	·475	·325	1·61	14·99	3·41	1·19
,, 14	,, ,, $\frac{1}{2}$	19·56	5·753	·475	·325	1·66	19·52	4·50	1·18
,, 14	,, ,, $\frac{5}{8}$	24·18	7·112	·475	·325	1·71	23·8	5·55	1·18
,, 14	,, ,, $\frac{3}{4}$	28·70	8·441	·475	·325	1·76	27·8	6·56	1·17
,, 14	,, ,, 1	37·41	11·003	·475	·325	1·85	35·09	8·46	1·16
,, 13	5 × 5 × $\frac{5}{16}$	10·30	3·028	·425	·300	1·34	7·18	1·96	·99
,, 13	,, ,, $\frac{3}{8}$	12·27	3·610	·425	·300	1·37	8·51	2·34	·98
,, 13	,, ,, $\frac{1}{2}$	16·15	4·750	·425	·300	1·42	11·0	3·07	·98
,, 13	,, ,, $\frac{5}{8}$	19·92	5·860	·425	·300	1·47	13·4	3·80	·98
,, 13	,, ,, $\frac{3}{4}$	23·59	6·938	·425	·300	1·51	15·5	4·44	·96
,, 12	$4\frac{1}{2}$ × $4\frac{1}{2}$ × $\frac{3}{8}$	11·00	3·236	·400	·275	1·22	6·14	1·87	·88
,, 12	,, ,, $\frac{1}{2}$	14·46	4·252	·400	·275	1·29	7·92	2·47	·87
,, 12	,, ,, $\frac{5}{8}$	17·80	5·236	·400	·275	1·34	9·56	3·03	·87
,, 12	,, ,, $\frac{3}{4}$	21·04	6·189	·400	·275	1·39	11·1	3·57	·87
,, 11	4 × 4 × $\frac{5}{16}$	8·17	2·402	·350	·250	1·10	3·61	1·24	·78
,, 11	,, ,, $\frac{3}{8}$	9·72	2·859	·350	·250	1·12	4·26	1·48	·78
,, 11	,, ,, $\frac{1}{2}$	12·75	3·749	·350	·250	1·17	5·46	1·93	·77
,, 11	,, ,, $\frac{5}{8}$	15·67	4·609	·350	·250	1·22	6·56	2·36	·77
,, 10	$3\frac{1}{2}$ × $3\frac{1}{2}$ × $\frac{5}{16}$	7·11	2·091	·325	·225	·975	2·39	·95	·68
,, 10	,, ,, $\frac{3}{8}$	8·45	2·485	·325	·225	1·00	2·80	1·12	·68
,, 10	,, ,, $\frac{1}{2}$	11·05	3·251	·325	·225	1·05	3·57	1·46	·68
,, 10	,, ,, $\frac{5}{8}$	13·55	3·985	·325	·225	1·09	4·27	1·77	·68

	1	2	3	4	5	6	7	8	9
B.S.E.A. 9	$3 \times 3 \times \frac{1}{4}$	4·90	1·44	·300	·200	·827	1·21	·56	·59
,, 9	,, ,, $\frac{5}{16}$	6·05	1·779	·300	·200	·853	1·47	·68	·58
,, 9	,, ,, $\frac{3}{8}$	7·18	2·111	·300	·200	·877	1·72	·81	·58
,, 9	,, ,, $\frac{1}{2}$	9·36	2·752	·300	·200	·924	2·19	1·05	·58
,, 9	,, ,, $\frac{5}{8}$	11·43	3·362	·300	·200	·970	2·59	1·28	·58
,, 7	$2\frac{1}{2} \times 2\frac{1}{2} \times \frac{1}{4}$	4·04	1·187	·275	·200	·703	·677	·38	·48
,, 7	,, ,, $\frac{5}{16}$	4·98	1·464	·275	·200	·728	·822	·46	·48
,, 7	,, ,, $\frac{3}{8}$	5·89	1·733	·275	·200	·752	·962	·55	·48
,, 7	,, ,, $\frac{1}{2}$	7·65	2·249	·275	·200	·799	1·21	·71	·48
,, 6	$2\frac{1}{4} \times 2\frac{1}{4} \times \frac{1}{4}$	3·61	1·063	·250	·175	·643	·489	·30	·44
,, 6	,, ,, $\frac{5}{16}$	4·45	1·309	·250	·175	·668	·592	·37	·43
,, 6	,, ,, $\frac{3}{8}$	5·26	1·547	·250	·175	·692	·686	·44	·43
,, 5	$2 \times 2 \times \frac{3}{16}$	2·43	·715	·250	·175	·554	·260	·18	·39
,, 5	,, ,, $\frac{1}{4}$	3·19	·938	·250	·175	·581	·336	·24	·39
,, 5	,, ,, $\frac{5}{16}$	3·92	1·153	·250	·175	·605	·401	·29	·38
,, 5	,, ,, $\frac{3}{8}$	4·62	1·36	·250	·175	·629	·467	·34	·38
,, 4	$1\frac{3}{4} \times 1\frac{3}{4} \times \frac{3}{16}$	2·11	·622	·225	·150	·495	·172	·14	·34
,, 4	,, ,, $\frac{1}{4}$	2·77	·814	·225	·150	·520	·220	·18	·34
,, 4	,, ,, $\frac{5}{16}$	3·39	·997	·225	·150	·544	·264	·22	·34
,, 3	$1\frac{1}{2} \times 1\frac{1}{2} \times \frac{3}{16}$	1·79	·526	·200	·150	·434	·105	·10	·29
,, 3	,, ,, $\frac{1}{4}$	2·33	·686	·200	·150	·458	·134	·13	·29
,, 3	,, ,, $\frac{5}{16}$	2·85	·839	·200	·150	·482	·159	·16	·29
,, 2	$1\frac{1}{4} \times 1\frac{1}{4} \times \frac{3}{16}$	1·47	·433	·200	·150	·371	·058	·07	·24
,, 2	,, ,, $\frac{1}{4}$	1·91	·561	·200	·150	·396	·073	·09	·23

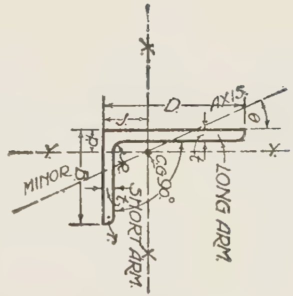
Note. The *least* radius of gyration (column 9) occurs about an axis which cuts the two arms and makes an angle of 45° with **XX** and **YY**.

The radius of gyration about any axis is the square root of "the moment of inertia about that axis divided by the area"

$$\text{or, } g = \sqrt{\frac{I}{A}}.$$

APPENDIX I (*cont.*)

(*e*) BRITISH STANDARD UNEQUAL ANGLES



DIMENSIONS AND PROPERTIES OF UNEQUAL ANGLES IN INCH UNITS

Reference Mark	Size and Thickness D × B × t (Inches)	Weight per Foot in lbs.	Area in Square Inches	Radii		Dimensions		Moments of Inertia		Section Moduli		Angle θ Degrees	Least Radius of Gyration about inclined axis
				Root R	Toe r	J	P	About XX	About YY	About XX	About YY		
	1	2	3	4	5	6	7	8	9	10	11	12	13
B.S.U.A. 25	7 × 3½ × ⅜	12·91	3·797	·425	·30	2·45	·713	19·30	3·32	4·24	1·19	15	·75
,, 25	,, ,, ½	17·00	5·000	·425	·30	2·50	·764	25·1	4·28	5·58	1·56	14½	·74
,, 25	,, ,, ⅝	20·98	6·172	·425	·30	2·55	·814	30·55	5·15	6·86	1·92	14½	·74
,, 25	,, ,, ¾	24·86	7·313	·425	·30	2·60	·862	35·68	5·95	8·11	2·26	14	·73
,, 24	6½ × 4½ × ⅜	13·54	3·982	·45	·325	2·03	1·04	17·08	6·76	3·82	1·95	25½	·98
,, 24	,, ,, ½	17·84	5·248	·45	·325	2·08	1·09	22·2	8·75	5·02	2·57	25	·97
,, 24	,, ,, ⅝	22·04	6·482	·45	·325	2·13	1·14	27·09	10·60	6·20	3·15	25	·96
,, 24	,, ,, ¾	26·13	7·686	·45	·325	2·18	1·19	31·66	12·32	7·33	3·72	25	·96

	1	2	3	4	5	6	7	8	9	10	11	12	13
B.S.U.A. 22	$6\frac{1}{2} \times 3\frac{1}{2} \times \frac{3}{8}$	12·27	3·610	·425	·30	2·22	·741	15·7	3·27	3·67	1·18	$16\frac{1}{2}$	·75
,, 22	,, ,, $\frac{1}{2}$	16·15	4·750	·425	·30	2·28	·792	20·4	4·20	4·83	1·55	$16\frac{1}{2}$	·75
,, 22	,, ,, $\frac{5}{8}$	19·92	5·860	·425	·30	2·33	·841	24·83	5·06	5·95	1·90	16	·74
,, 21	$6 \times 4 \times \frac{3}{8}$	12·27	3·610	·425	·30	1·91	·923	13·2	4·73	3·23	1·54	$23\frac{1}{2}$	·87
,, 21	,, ,, $\frac{1}{2}$	16·15	4·750	·425	·30	1·96	·974	17·1	6·10	4·23	2·02	$23\frac{1}{2}$	·86
,, 21	,, ,, $\frac{5}{8}$	19·92	5·860	·425	·30	2·02	1·02	20·8	7·36	5·23	2·47	$23\frac{1}{2}$	·86
,, 21	,, ,, $\frac{3}{4}$	23·59	6·938	·425	·30	2·06	1·07	24·24	8·52	6·15	2·91	23	·85
,, 20	$6 \times 3\frac{1}{2} \times \frac{3}{8}$	11·64	3·424	·40	·275	2·01	·773	12·6	3·22	3·16	1·18	19	·76
,, 20	,, ,, $\frac{1}{2}$	15·31	4·502	·40	·275	2·06	·823	16·4	4·14	4·16	1·55	19	·75
,, 20	,, ,, $\frac{5}{8}$	18·87	5·549	·40	·275	2·11	·872	19·88	4·97	5·11	1·89	$18\frac{1}{2}$	·75
,, 20	,, ,, $\frac{3}{4}$	22·32	6·564	·40	·275	2·16	·919	23·14	5·74	6·03	2·22	18	·74
,, 19	$5\frac{1}{2} \times 3\frac{1}{2} \times \frac{3}{8}$	11·00	3·236	·40	·275	1·80	·807	9·93	3·15	2·68	1·17	22	·76
,, 19	,, ,, $\frac{1}{2}$	14·46	4·252	·40	·275	1·85	·857	12·80	4·05	3·51	1·53	22	·75
,, 19	,, ,, $\frac{5}{8}$	17·80	5·236	·40	·275	1·90	·905	15·6	4·86	4·33	1·87	$21\frac{1}{2}$	·75
,, 18	$5\frac{1}{2} \times 3 \times \frac{5}{16}$	8·71	2·562	·375	·25	1·87	·636	8·00	1·72	2·20	·73	17	·65
,, 18	,, ,, $\frac{3}{8}$	10·37	3·050	·375	·25	1·90	·662	9·45	2·02	2·62	·86	17	·64
,, 18	,, ,, $\frac{1}{2}$	13·61	4·003	·375	·25	1·95	·711	12·2	2·58	3·44	1·13	$16\frac{1}{2}$	·64
,, 18	,, ,, $\frac{5}{8}$	16·74	4·925	·375	·25	2·00	·759	14·7	3·08	4·20	1·37	$16\frac{1}{2}$	·63
,, 17	$5 \times 4 \times \frac{3}{8}$	11·00	3·236	·40	·275	1·51	1·01	7·96	4·53	2·28	1·52	32	·85
,, 17	,, ,, $\frac{1}{2}$	14·46	4·252	·40	·275	1·56	1·06	10·3	5·82	2·99	1·98	32	·84
,, 17	,, ,, $\frac{5}{8}$	17·80	5·236	·40	·275	1·60	1·11	12·4	7·01	3·66	2·43	32	·83
,, 16	$5 \times 3\frac{1}{2} \times \frac{5}{16}$	8·71	2·562	·375	·25	1·56	·822	6·47	2·63	1·88	·98	$25\frac{1}{2}$	·76
,, 16	,, ,, $\frac{3}{8}$	10·37	3·050	·375	·25	1·59	·848	7·64	3·09	2·24	1·17	$25\frac{1}{2}$	·75
,, 16	,, ,, $\frac{1}{2}$	13·61	4·003	·375	·25	1·64	·897	9·86	3·96	2·93	1·52	$25\frac{1}{2}$	·75
,, 16	,, ,, $\frac{5}{8}$	16·74	4·925	·375	·25	1·69	·944	11·9	4·75	3·60	1·86	25	·74

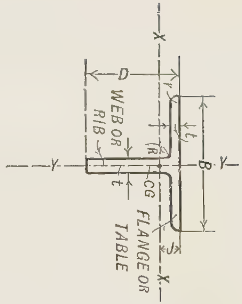
APPENDIX I (*cont.*)

Reference Mark	Size and Thickness $D \times B \times t$ (Inches)	Weight per Foot in lbs.	Area in Square Inches	Radii		Dimensions		Moments of Inertia		Section Moduli		Angle θ Degrees	Least Radius of Gyration about inclined axis
				Root R	Toe r	J	P	About XX	About YY	About XX	About YY		
	1	2	3	4	5	6	7	8	9	10	11	12	13
B.S.U.A. 15	$5 \times 3 \times \frac{5}{16}$	8·17	2·402	·35	·25	1·66	·667	6·14	1·68	1·84	·72	20	·65
,, 15	,, ,, $\frac{3}{8}$	9·72	2·859	·35	·25	1·68	·693	7·24	1·97	2·18	·85	$19\frac{1}{2}$	·65
,, 15	,, ,, $\frac{1}{2}$	12·75	3·749	·35	·25	1·73	·742	9·33	2·51	2·85	1·11	$19\frac{1}{2}$	·64
,, 15	,, ,, $\frac{5}{8}$	15·67	4·609	·35	·25	1·78	·789	11·25	3·00	3·49	1·36	19	·64
,, 14	$4\frac{1}{2} \times 3\frac{1}{2} \times \frac{5}{16}$	8·17	2·402	·35	·25	1·36	·866	4·82	2·55	1·54	·97	$30\frac{1}{2}$	·74
,, 14	,, ,, $\frac{3}{8}$	9·72	2·859	·35	·25	1·39	·891	5·69	3·00	1·83	1·15	$30\frac{1}{2}$	·74
,, 14	,, ,, $\frac{1}{2}$	12·75	3·749	·35	·25	1·44	·940	7·31	3·84	2·39	1·5	30	·74
,, 14	,, ,, $\frac{5}{8}$	15·67	4·609	·35	·25	1·48	·987	8·81	4·61	2·92	1·83	30	·74
,, 13	$4\frac{1}{2} \times 3 \times \frac{5}{16}$	7·64	2·250	·35	·25	1·44	·703	4·58	1·63	1·50	·711	$23\frac{1}{2}$	·65
,, 13	,, ,, $\frac{3}{8}$	9·08	2·670	·35	·25	1·47	·728	5·40	1·92	1·78	·842	$23\frac{1}{2}$	·64
,, 13	,, ,, $\frac{1}{2}$	11·90	3·500	·35	·25	1·52	·777	6·93	2·44	2·33	1·10	23	·64
,, 13	,, ,, $\frac{5}{8}$	14·61	4·300	·35	·25	1·57	·824	8·34	2·91	2·85	1·34	23	·64
,, 12	$4 \times 3\frac{1}{2} \times \frac{5}{16}$	7·64	2·246	·35	·25	1·16	·915	3·46	2·47	1·22	·96	37	·72
,, 12	,, ,, $\frac{3}{8}$	9·08	2·671	·35	·25	1·19	·941	4·08	2·90	1·45	1·13	37	·72
,, 12	,, ,, $\frac{1}{2}$	11·90	3·499	·35	·25	1·24	·990	5·23	3·71	1·89	1·48	37	·71
,, 12	,, ,, $\frac{5}{8}$	14·61	4·296	·35	·25	1·28	1·04	6·28	4·44	2·31	1·80	$36\frac{1}{2}$	·71
,, 11	$4 \times 3 \times \frac{5}{16}$	7·11	2·091	·325	·225	1·24	·746	3·31	1·59	1·20	·71	$28\frac{1}{2}$	·64
,, 11	,, ,, $\frac{3}{8}$	8·45	2·485	·325	·225	1·27	·771	3·89	1·87	1·42	·84	$28\frac{1}{2}$	·64
,, 11	,, ,, $\frac{1}{2}$	11·05	3·251	·325	·225	1·31	·819	4·98	2·37	1·85	1·09	$28\frac{1}{2}$	·63
,, 11	,, ,, $\frac{5}{8}$	13·55	3·985	·325	·225	1·36	·865	5·96	2·83	2·26	1·33	28	·63

	1	2	3	4	5	6	7	8	9	10	11	12	13
B.S.U.A. 10	4 × 2½ × ¼	5·31	1·563	·325	·225	1·30	·561	2·54	·767	·94	·40	21	·54
,, 10	,, ,, 5/16	6·58	1·934	·325	·225	1·33	·587	3·11	·935	1·16	·49	21	·54
,, 10	,, ,, ⅜	7·81	2·298	·325	·225	1·35	·612	3·65	1·09	1·38	·58	21	·53
,, 10	,, ,, ½	10·20	3·001	·325	·225	1·40	·660	4·66	1·38	1·79	·75	20½	·53
,, 9	3½ × 3 × 5/16	6·58	1·934	·325	·225	1·04	·792	2·27	1·53	·92	·69	35½	·62
,, 9	,, ,, ⅜	7·81	2·298	·325	·225	1·07	·819	2·67	1·80	1·10	·83	35½	·62
,, 9	,, ,, ½	10·20	3·001	·325	·225	1·11	·867	3·40	2·28	1·42	1·07	35½	·61
,, 9	,, ,, ⅝	12·49	3·673	·325	·225	1·16	·912	4·05	2·71	1·73	1·30	35	·61
,, 8	3½ × 2½ × ¼	4·90	1·44	·30	·20	1·10	·602	1·76	·748	·73	·39	26½	·54
,, 8	,, ,, 5/16	6·05	1·779	·30	·20	1·12	·627	2·15	·910	·90	·49	26½	·54
,, 8	,, ,, ⅜	7·18	2·111	·30	·20	1·15	·652	2·52	1·06	1·07	·57	26	·53
,, 8	,, ,, ½	9·36	2·752	·30	·20	1·20	·699	3·20	1·34	1·39	·74	26	·53
,, 7	3 × 2½ × ¼	4·46	1·312	·275	·20	·895	·648	1·14	·716	·54	·39	34	·52
,, 7	,, ,, 5/16	5·51	1·62	·275	·20	·921	·673	1·39	·871	·67	·48	34	·52
,, 7	,, ,, ⅜	6·53	1·921	·275	·20	·945	·697	1·62	1·02	·79	·57	34	·52
,, 7	,, ,, ½	8·50	2·499	·275	·20	·992	·744	2·05	1·28	1·02	·73	33½	·52
,, 6	3 × 2 × ¼	4·04	1·187	·275	·20	·976	·482	1·06	·373	·52	·25	23½	·43
,, 6	,, ,, 5/16	4·98	1·464	·275	·20	1·00	·508	1·29	·452	·65	·30	23	·42
,, 6	,, ,, ⅜	5·89	1·733	·275	·20	1·03	·532	1·50	·525	·76	·36	23	·42
,, 6	,, ,, ½	7·65	2·249	·275	·20	1·07	·578	1·89	·656	·98	·46	22½	·42
,, 5	2½ × 2 × ¼	3·61	1·063	·25	·175	·774	·527	·636	·359	·37	·24	32	·42
,, 5	,, ,, 5/16	4·45	1·309	·25	·175	·799	·552	·770	·433	·45	·30	31½	·42
,, 5	,, ,, ⅜	5·26	1·547	·25	·175	·823	·575	·895	·502	·53	·35	31½	·42
,, 4	2 × 1½ × 3/16	2·11	·622	·225	·150	·627	·381	·240	·115	·17	·10	28½	·32
,, 4	,, ,, ¼	2·77	·814	·225	·150	·653	·407	·308	·146	·23	·13	28	·31
,, 4	,, ,, 5/16	3·39	·997	·225	·150	·678	·431	·369	·174	·28	·16	28	·31

APPENDIX I (*cont.*)

(*f*) BRITISH STANDARD TEES



DIMENSIONS AND PROPERTIES OF TEES IN INCH UNITS

Reference Mark	Size and Thickness B × D × t (Inches)	Weight per Foot in lbs.	Area in Square Inches	Radii		Dimension	Moments of Inertia		Section Moduli		Radii of Gyration	
				Table Root R	Table Toe r	J	About XX	About YY	About XX	About YY	About XX	About YY
	1	2	3	4	5	6	7	8	9	10	11	12
B.S.T. 21	6 × 4 × $\frac{3}{8}$	12·36	3·634	·425	·300	·915	4·700	6·344	1·52	2·11	1·137	1·321
,, 21	,, ,, $\frac{1}{2}$	16·22	4·771	·425	·300	·968	6·070	8·621	2·00	2·87	1·128	1·344
,, 21	,, ,, $\frac{5}{8}$	19·99	5·878	·425	·300	1·02	7·350	10·912	2·47	3·64	1·118	1·362
,, 20	6 × 3 × $\frac{1}{2}$	14·53	4·272	·400	·275	·684	2·635	8·649	1·14	2·88	·785	1·423
,, 20	,, ,, $\frac{5}{8}$	17·87	5·256	·400	·275	·732	3·144	10·938	1·39	3·65	·773	1·443
,, 19	5 × 4 × $\frac{3}{8}$	11·07	3·257	·400	·275	·998	4·471	3·691	1·49	1·48	1·172	1·065
,, 19	,, ,, $\frac{1}{2}$	14·51	4·268	·400	·275	1·05	5·772	5·017	1·96	2·01	1·163	1·084
,, 17	5 × 3 × $\frac{3}{8}$	9·78	2·875	·350	·250	·691	1·973	3·716	·85	1·49	·828	1·137
,, 17	,, ,, $\frac{1}{2}$	12·79	3·762	·350	·250	·741	2·516	5·031	1·11	2·01	·818	1·156
,, 15	4 × 4 × $\frac{3}{8}$	9·77	2·872	·350	·250	1·11	4·189	1·901	1·45	·95	1·208	·814
,, 15	,, ,, $\frac{1}{2}$	12·78	3·758	·350	·250	1·16	5·402	2·590	1·90	1·29	1·199	·830

	1	2	3	4	5	6	7	8	9	10	11	12
B.S.T. 14	$4 \times 3 \times \frac{3}{8}$	8·49	2·498	·325	·225	·767	1·860	1·914	·83	·96	·863	·875
,, 14	,, ,, $\frac{1}{2}$	11·08	3·262	·325	·225	·816	2·365	2·599	1·08	1·30	·851	·893
,, 13	$3\frac{1}{2} \times 3\frac{1}{2} \times \frac{3}{8}$	8·49	2·496	·325	·225	·988	2·768	1·284	1·10	·73	1·053	·717
,, 13	,, ,, $\frac{1}{2}$	11·08	3·259	·325	·225	1·04	3·543	1·752	1·44	1·00	1·043	·733
,, 11	$3 \times 3 \times \frac{3}{8}$	7·21	2·121	·300	·200	·868	1·708	·816	·80	·54	·897	·620
,, 11	,, ,, $\frac{1}{2}$	9·38	2·76	·300	·200	·918	2·165	1·115	1·04	·74	·886	·636
,, 10	$3 \times 2\frac{1}{2} \times \frac{3}{8}$	6·56	1·929	·275	·200	·695	1·015	·814	·56	·54	·725	·650
,, 10	,, ,, $\frac{1}{2}$	8·52	2·506	·275	·200	·742	1·275	1·109	·73	·74	·713	·665
,, 8	$2\frac{1}{2} \times 2\frac{1}{2} \times \frac{1}{4}$	4·07	1·197	·275	·200	·697	·677	·302	·38	·24	·752	·502
,, 8	,, ,, $\frac{5}{16}$	5·01	1·474	·275	·200	·724	·823	·387	·46	·31	·747	·512
,, 8	,, ,, $\frac{3}{8}$	5·92	1·742	·275	·200	·750	·959	·473	·55	·38	·742	·521
,, 7	$2\frac{1}{4} \times 2\frac{1}{4} \times \frac{1}{4}$	3·64	1·071	·250	·175	·638	·488	·224	·30	·20	·675	·457
,, 7	,, ,, $\frac{3}{8}$	5·28	1·554	·250	·175	·689	·685	·349	·44	·31	·664	·474
,, 6	$2 \times 2 \times \frac{1}{4}$	3·22	·947	·250	·175	·579	·337	·157	·24	·16	·597	·407
,, 6	,, ,, $\frac{3}{8}$	4·64	1·367	·250	·175	·628	·469	·246	·34	·25	·586	·424
,, 5	$1\frac{1}{2} \times 2 \times \frac{1}{4}$	2·79	·820	·225	·150	·648	·307	·068	·23	·09	·612	·288
,, 5	,, ,, $\frac{5}{16}$	3·41	1·003	·225	·150	·674	·369	·088	·28	·12	·607	·296
,, 4	$1\frac{3}{4} \times 1\frac{3}{4} \times \frac{1}{4}$	2·79	·820	·225	·150	·519	·221	·107	·18	·12	·520	·361
,, 4	,, ,, $\frac{5}{16}$	3·40	·999	·225	·150	·544	·265	·137	·22	·16	·515	·370
,, 3	$1\frac{1}{2} \times 1\frac{1}{2} \times \frac{3}{16}$	1·81	·531	·200	·150	·435	·106	·048	·10	·06	·447	·301
,, 3	,, ,, $\frac{1}{4}$	2·35	·692	·200	·150	·460	·135	·067	·13	·09	·442	·312

APPENDIX II

LIST OF SYMBOLS

General

A	Total area.
a	Area less than A.
B	Bending moment in beams due to external loads.
B_f	Fixing bending moment (see Fixed beams).
B_g	Bending moment at any section G, etc.
$B_{\text{max.}}$	Maximum bending moment.
b	Breadth.
C	Total compressive force.
c	Compressive stress.
c_b	Compressive or bearing stress (bolts and rivets).
c_c	Crushing strength (see Columns).
C.G.	Centre of gravity.
d	Depth or diameter.
E	Modulus of Elasticity or Young's Modulus for any material (E_s for steel, and E_c for concrete, etc.). Equilibrant force.
e	Eccentricity of a force.
F	A force. Factor of safety. Working or Reduction factor.
f	Flexural or bending stress.
f_c	Maximum compressive bending stress.
f_t	Maximum tensile bending stress.
f_d	Difference between two given stresses, "stress-difference".
G	Shear modulus. Centroid of an area.
g	Gyration radius ($= \sqrt{I/A}$).
H	Horizontal component of a force. Horizontal thrust in an arch.
I	Moment of inertia.
I_e	Equivalent moment of inertia.
I_x	Moment of inertia about an axis X–X, etc.
L	A length. Effective length or span of a beam.
l	Any length shorter than L. Length of a column with free ends.
M	Moment of a force (not in beams).
m	A Constant (see Columns).
P	Total push or pull. Buckling load on a column.
p	Pressure or stress due to a direct push or pull. Pitch (of rivets or stirrups).
R	Internal moment of resistance of a beam. Radius of curvature. Resistance of a single rivet or stirrup.
R_a, R_b, etc.	Reactions at the supports of a beam.
S	Total vertical shear force at a section of a beam (also S_a, $S_{\text{max.}}$, etc.).
s	Shear stress.
s_m	Mean shear stress ($= S/bd$).
T	Total tensile force.
t	Tensile stress.
th	Thickness.
V	Vertical component of a force.

W Weight or load (total or concentrated).
w Weight per unit of length, or per unit of volume (density).
y Distance from neutral axis to a given layer in a beam (usually to the outer layers in symmetrical sections).
y_c Distance from neutral axis to compression edge.
y_t Distance from neutral axis to tension edge.
Z Section modulus.

Additional Special Symbols

Reinforced concrete

A_e Equivalent area.
A_c Area of compression reinforcement.
A_t Area of tension reinforcement.
a Lever arm, distance between centre of tension reinforcement and centre of compression forces.
b Breadth of rectangular beam. Effective breadth of slab (see T-beams).
b_r Breadth of rib.
c Compressive stress in beams (usually at compression edge). Compressive stress in reinforced concrete columns.
D Outside diameter of column.
d Effective depth of beam.
d_1, d_2 Effective diameters or lateral dimensions of columns.
d_s Total depth of slab.
m Modular ratio ($= E_s/E_c$).
n Depth to neutral axis from compression edge.
r Ratio of area of tensile reinforcement to area of concrete.

Timber construction

B.W. Central breaking weight (in beams).
c Safe end grain compressive stress.
c_s Safe side grain compressive stress.
c_θ Safe compressive stress on surface inclined at angle θ to grain.
f_r Modulus of rupture.
s Longitudinal shear stress.
s_b Longitudinal shear stress in beams.

Miscellaneous

δ (*delta*) Deflection, usually maximum deflection.
θ (*theta*) Inclination to the horizontal plane.
μ (*mu*) Coefficient of friction.
π (*pi*) Ratio of circumference to diameter of a circle ($\doteq 3{\cdot}1416$).
Σ (*sigma*) Sign of algebraic summation.
ϕ (*phi*) The angle of friction between two materials. The angle of repose for a material.
Ton In this volume the term "ton" refers to the measure of weight used in Great Britain of 2240 lbs. The American "ton" (sometimes "short" or "net ton") contains 2000 lbs. Confusion between the two units may be avoided by expressing working stresses in lbs. to the nearest 1000 lbs.

ANSWERS TO PROBLEMS

PROBLEMS I, p. 10.

1. $SO = 10{,}000$ lbs.; $OT = 8660$ lbs., away from O.
2. Jib, 17·32 tons; tie, 10 tons.
3. 1·93 tons.
4. NO, 4826 lbs. (Sense, read "from N to O"); PO, 985 lbs.
5. NO, 4570 lbs.; PO, 981 lbs.

PROBLEMS II, p. 15.

1. $R_A = 7{\cdot}1$ tons; $R_B = 9{\cdot}9$ tons.
2. $R_A = 1100$ lbs.; $R_B = 2200$ lbs.
3. $V_B = 93\frac{3}{4}$ lbs.; $H_B = 62\frac{1}{2}$ lbs.
4. $R_C = 150$ lbs.; vertical component = 300 lbs. and horizontal component = 150 lbs.
5. $R_B = 8\frac{1}{3}$ tons; $R_C = 13\frac{1}{3}$ tons.
6. $R_A = 133\frac{1}{3}$ lbs.; R_B (vert.) = 2000 lbs., R_B (hori.) = $133\frac{1}{3}$ lbs.

PROBLEMS III, p. 24.

2. 50,400 lbs.; 2·083 ft. from left.
3. 6·93 ft. from lower right-hand corner.
4. 7·596 ft. from lower right-hand corner.
5. 3·96 ins. from right-hand side.
6. 3·4 ins. from base.
7. 3·96 ins. from right and 1·03 ins. from base.
8. 36·6 ins. from right-hand side.

PROBLEMS IV, p. 40.

1. $R_A = 7{\cdot}1$ tons; $R_B = 9{\cdot}9$ tons.
2. $R_A = 1100$ lbs.; $R_B = 2200$ lbs.
3. 16 lbs., acting 3·47 ins. (nearly) from force AB.
4. $R_M = 1630$ lbs.; $R_N = 1950$ lbs.; both act at 76° to horizontal.
5. R_M (vertical) = 1630 lbs.; $R_N = 2130$ lbs., at 66° to the horizontal.

PROBLEMS V, p. 62. No answers.

PROBLEMS VI, p. 83.

1. $E = 30$ lbs., acting within triangle at 19 ins. from 8 lbs. wt., 17·2 ins. from 10 lbs. wt. and 15·5 ins. from 12 lbs. wt.
2. $E = 14$ lbs., acting in same direction as 8 lbs., and outside the triangle at 19·5 ins. from 12 lbs. wt. and 22·8 ins. from 10 lbs. wt.
3. At A and B, $\frac{1}{5}$ ton downwards, $\frac{1}{2}$ ton upwards; at C and D, $\frac{1}{2}$ ton upwards.
4. $W_{\max.} = 2000$ lbs.
5. Position 1: $R_A = R_B = 3{\cdot}08$ tons (downwards), $R_C = 10{\cdot}16$ tons (upwards); Position 6: $R_A = 5{\cdot}33$ tons (upwards), $R_B = 0$, $R_C = 1{\cdot}33$ tons (downwards).

PROBLEMS VII, p. 119.

1. (*a*) Forces in A and C, each 8·66 lbs.; (*b*) Force in mast, 5 lbs.
2. 25·98 lbs. **3.** PC = 1154 lbs.; PA = 577 lbs.
4. PD = 500 lbs.; PA = 866 lbs.
5. PD = 707 lbs.; PA = 1225 lbs.
6. (*a*) 2·3 tons; (*b*) 7·3 tons and 2·5 tons; (*c*) 3·43 tons; (*d*) 3·82 tons.
7. Force in each stay, 4830 lbs; $R_D = R_F$ = 4330 lbs.; R_E = 8660 lbs.
8. Force in AE = 2·4 tons; force in FE = 3·18 tons.

PROBLEMS VIII, p. 133.

1. S = 10 lbs.; $T = C$ = 15 lbs. **2.** 48 lb. ins.; 4 ins.
3. 1000 lbs. acting upwards; anti-clockwise couple of 60,000 lb. ins.
4. + 16 lbs.; − 120 lb. ins. **5.** 16 lbs. at 7·5 ins. from a.

PROBLEMS IX, p. 155.

1. 8960 lbs.; 376,320 lb. ins.
2. 8000 lbs.; 320,000 lb. ins.
3. 2500 lbs.; 300,000 lb. ins.
4. 56,250 lb. ft.; 5000 lbs. and 50,000 lb. ft.
5. 24 ton ft. **6.** 30,240 lb. ft.
7. $B_A = B_B$ = 360,000 lb. ins.; B_C = 158,400 lb. ins.
8. Distance between A and B should be 12·9 ft.
9. $R_A = 6\frac{9}{11}$ tons; $R_B = 5\frac{2}{11}$ tons; $B_C = 27\frac{3}{11}$ ton ft.; $B_D = 38\frac{2}{11}$ ton ft.; $B_E = 31\frac{1}{11}$ ton ft.
10. 20,000 lb. ft. to an inch.

PROBLEMS X, p. 165.

1. 176 ton ft.; $7\frac{2}{7}$ ft. from A.
2. 30 lb. per ft.; 3375 lbs. and 16,875 lb. ft.
3. 8·66 ft. from B; 6480 lb. ft.

PROBLEMS XI, p. 175.

1. Force in AB, 14 lbs.; AE, 39·6 lbs.; FE, 42 lbs.
2. Force in BE, 14 lbs.; AF, 28 lbs.; AE, 39·6 lbs.
3. Force in AF, 12,000 lbs.; EB, 4000 lbs.; EF, 13,333 lbs.; DE, 0; FB, 17,900 lbs.
4. Force in GF, $5\frac{1}{2}$ tons, compression; GD, 2·12 tons, comp.; GC, 4 tons, tension; GB, 3·535 tons, comp.; GH, $4\frac{1}{2}$ tons, comp.
5. Force in CD, 99,000 lbs., comp.; DH, 45,000 lbs., comp.; HK, 81,000 lbs., tension; CK, 33,000 lbs., comp.; CH, 25,450 lbs., tension; KD, no stress.
6. Force in AB, 14,580 lbs., comp.; BC, 3500 lbs., tension; AC, 12,500 lbs., tension; CD, 4860 lbs. comp.
7. Compressive force in DE, 1000 lbs.; in CF, 2000 lbs., and in BE, 3000 lbs. Tensile force in CE, 1230 lbs., in BF, 2460 lbs., and in AG, 3690 lbs.

PROBLEMS XII, p. 191.

1. (*a*) 5·625 sq. ins.; (*b*) 2·18 ins. **2.** 0·128 in.
3. (*a*) 29,166,000 lbs. per sq. in.; (*b*) 30 tons per sq. in.; (*c*) 54 %; (*d*) 18·75 %.
4. 65 tons (approx.).
5. (*a*) 19,800 lbs. per sq. in., tension; (*b*) 12,000 lbs. per sq. in., compression.
6. (*a*) Steel, 9000 lbs. per sq. in.; concrete, 600 lbs. per sq. in.; (*b*) 13,500 lbs. by the steel, and 60,000 lbs. by the concrete; (*c*) total load 73,500 lbs., or 32·8 tons.

PROBLEMS XIII, p. 207.

1. 7500 ins. **2.** 12,000 lbs. per sq. in., and 4000 lbs. per sq. in.
3. 224,000 lbs. per sq. in.
4. 0·00533 in.; 0·004 in.; and 0·00133 in. **5.** 5·83 tons.
6. (*a*) 30·77 inch units³, and 57 inch units³ (approx.); (*b*) 85·7 ton in.; (*c*) 2·785 tons per sq. in.

PROBLEMS XIV, p. 221.

1. 332·75 inch units⁴, and 60·5 inch units³.
2. 2688 lbs. **3.** 37·4 inch units³. **4.** $\frac{bd^3}{12}$.
5. 266 inch units⁴ (approx.). **6.** 135 lbs. per sq. ft.

PROBLEMS XV, p. 239.

1. 166·6 lbs. per sq. in. **2.** 105 lbs. per sq. in.; 70 lbs. per sq. in.
3. 2666 lbs. **4.** 55 ins. for a central load.
5. 18·75 ins. and 1920 lbs. **6.** 4·375 tons.
7. 62·5 lbs. per sq. in. **8.** 1·43 ton per sq. in. **9.** 1·43 ton per sq. in.

PROBLEMS XVI, p. 254.

1. 0·81 in. **2.** 0·027 radian. **3.** 4112 lbs.
4. 0·00394 radian, and 0·00251 radian.
5. Deflection at C = 0·333 in.;

$$\text{deflection at } E = 0{\cdot}438 \text{ in.} = \frac{\text{length}}{822} = 0{\cdot}00122\, L.$$

6. 0·00237 radian; 0·2194 in. **7.** 0·1408 in.

PROBLEMS XVII, p. 265.

1. $W_1 = \frac{W}{2}$. **2.** 12·5 ton ft., and − 12·5 ton ft.
3. 0·1792 in. **4.** 1·5 ton ft., and − 3 ton ft.
5. Required value of I is 4·834 inch units⁴. The section having the nearest value is a 4½ in. by 2 in. new B.S.B. (I = 6·65 inch units⁴).
6. Negative bending moments at A and B are 67,500 lb. in.; positive bending moments between C and D are 40,500 lb. in.
7. $R_A = R_B$ = 900 lbs.; R_C = 3000 lbs.; B_C = − 43,200 lb. in; $B_{\max.}$ = 24,300 lb. in., and occurs at points 4·5 ft. from both A and B.
8. The value of the required Z is 2·57 inch units³; the B.S.B. 4¾ in. by 1¾ in. gives Z = 2·833 inch units³.
9. R_B = 4500 lbs.; B_C = − 216,000 lb. in.; $B_{\max.}$ = 121,500 lb. in., and occurs at point 4·5 ft. from B.

PROBLEMS XVIII, p. 285.

1. Thickness, $\frac{3}{8}$ in.; width, 2 ins.; 3·3 tons.
2. Use 2 rivets; thickness of bar $\frac{5}{8}$ in.; width (3·165 ins.) say $3\frac{1}{2}$ ins.; efficiency 70·7 %.
3. Use $\frac{5}{8}$ in. bar, 8 ins. wide; 6 rivets; $\frac{1}{2}$ in. cover plates.
4. 1000 inch units^4 (approx.); pitch 3 ins.
5. 5249 inch inits^4; 389 inch units^3.
6. W (dist.) = 80 tons (nearly); web $\frac{7}{16}$ in. or $\frac{1}{2}$ in. (latter preferable); pitch of rivets (checking for both web and flange rivets) 3 ins. (using four rows of rivets in flanges, two rivets on each section).
7. Based on simple check suggested at end of para. 164: 4 rivets in web of girder and 4 rivets in web of secondary girder.

PROBLEMS XIX, p. 307.

1. 2240 lbs. per sq. in.; 3422 lbs. per sq. in.; 3733 lbs. per sq. in.; 4978 lbs. per sq. in.
2. 21,180 lbs. (when max. stress in steel = 16,000 lbs. per sq. in., max. stress in timber will be 960 lbs. per sq. in.).
3. 0·012 in. **4.** A_t = 0·945 sq. in.; R = 186,200 lb. in.
5. R = 204,000 lb. in. (approx.); t = 13,500 lbs. per sq. in.
6. R = 155,000 lb. in. (approx.); c = 537 lbs. per sq. in.
7. d = 13·3 ins., say 14 ins.; A_t = 0·9 sq. in. (Use either three $\frac{5}{8}$ in. or five $\frac{1}{2}$ in. round bars.)
8. A_t = 1·25 sq. in.; t = 16,000 lbs. per sq. in.; c = 500 lbs. per sq. in.

PROBLEMS XX, p. 328.

1. R = 810,000 lb. in.; t = 16,000 lbs. per sq. in.; c = 490 lbs. per sq. in.
2. I_e = 9280 inch units^4; R = 824,700 lb. in.
3. A_t = 4·27 sq. in.
4. I_e = 2857 inch units^4; R = 297,000 lb. in.; c = 600 lbs. per sq. in. (approx.); stress in comp. reinforcement, 6000 lbs. per sq. in.
5. A_t = 3·08 sq. in.; A_c = 2·15 sq. in.
6. For assumed conditions grip stress = 93 lbs. per sq. in.
7. 183 lbs. per sq. in.; 13,800 lbs. **8.** 5·82 ins.

PROBLEMS XXI, p. 350.

1. (*a*) $8\frac{1}{3}$; (*b*) $12\frac{1}{2}$; (*c*) 50, all in sq. ft.
2. Maximum, 2·494 tons per sq. ft.; minimum, 1·066 ton per sq. ft.
3. e = 1·19 ft.; P = 51 tons.
4. Total pressure = 3942 lbs., or 29·2 lbs. per sq. ft.
5. From vertical face: at A–A, 1·25 ft. (nearly); at B–B, 2·5 ft. (nearly); at C–C, 3 ft. (nearly).
6. From vertical face and at each level: Empty, 2·1 ft., 2·79 ft., 3·49 ft., and 4·2 ft.; Full, 2·45 ft., 3·86 ft., 5·47 ft., and 7·2 ft.
7. From vertical face, 5·14 ft.; maximum pressure, 3116 lbs. per sq. ft.; minimum pressure, 243 lbs. per sq. ft.; horizontal thrust, 5120 lbs.
8. 4·15 ft.

PROBLEMS XXII, p. 369.

1. $H = 30$ lbs.; distance $= 9$ ins.
2. A suitable line can be drawn which is 10 ins. below the crown and 12 ins. from the lower corner of the abutment. (Other lines are also possible.)
3. A line can be drawn which passes near the centres of the crown and abutment joints and nowhere approaches the middle-third lines very closely.
4. $H = 8\frac{3}{4}$ lbs.; $V_a = 15\frac{3}{4}$ lbs.; $V_b = 19\frac{1}{4}$ lbs.
6. $W = 132{,}670$ lbs.; 24,330 lbs. per sq. in.

PROBLEMS XXIII, p. 390.

1. Total strain, 0·00354 in.; 590 lbs. per sq. in.
2. (*a*) 145·4; (*b*) 97. **3.** 1158 lbs.
4. (*a*) 12 tons (nearly); (*b*) 14·4 tons (approx.).
5. (*a*) 69·7 tons; (*b*) 50·8 tons.

PROBLEMS XXIV, p. 407.

1. 3·93 tons per sq. in., compression; 1·03 ton per sq. in., tension.
2. 26·8 tons (with stress not exceeding 10,000 lbs. per sq. in.).
3. 52,268 lbs., or $23\frac{1}{2}$ tons, nearly.
4. (See Fig. 273 (ii).) Max. stress at corner (1), 3·42 tons per sq. in.; minimum stress at corner (4), 0·24 ton per sq. in.
5. 50·4 tons.; 7500 lbs. per sq. in. **6.** 12 tons nearly.
7. (*a*) 3·25 ins.; (*b*) 56,200 lbs. or 25·2 tons; (*c*) 7500 lbs. per sq. in.
8. Assuming base plate 21 ins. square, the total magnitude of the "Z" required = 108 inch units^3; use 9 B.S.B.s 8 in. by 4 in. by 18 lbs. (total $Z = 13{\cdot}9 \times 9 = 125{\cdot}1$ inch units^3).
   (For top tier use 3 B.S.B.s 12 in. × 5 in. × 30 lbs., or 13 in. × 5 in. × 35 lbs.)
9. Area of reinforcement required in width of 10 ft. is 5·17 sq. in. Nineteen $\frac{5}{8}$ in. round bars ($A = 0{\cdot}306$ sq. in.) may be used, spaced at 6 in. centres.

PROBLEMS XXV, p. 438.

1. (*a*) $1\frac{1}{2}$ in. by 0·427 in. (say $\frac{1}{2}$ in.); (*b*) 4·22 in. (say $4\frac{1}{2}$ in.); (*c*) 8·44 in. (say $8\frac{1}{2}$ in.).
2. 4·88 tons. **3.** 5040 lbs. or 2·25 tons.
4. 32,400 lbs. or 10·45 tons.
5. (*a*) 9000 lb. in.; (*b*) 1060 lbs. per sq. in., tension.
6. $1\frac{7}{8}$ in.; 11 in.
7. $2\frac{1}{2}$ in.; $10\frac{1}{2}$ in. (allowing for reduction of area due to tenon between A and B).
8. (*a*) $2\frac{1}{2}$ in.; (*b*) $14\frac{1}{2}$ in.; (*c*) 1·03 in. (say $1\frac{1}{4}$ in.); (*d*) 0·925 in. (say 1 in.).
9. (*a*) $1\frac{5}{8}$ in.; (*b*) $14\frac{1}{2}$ in.; (*c*) 0·42 in. (say $\frac{7}{16}$ in. or $\frac{1}{2}$ in.).
10. 0·642 in. (say $\frac{3}{4}$ in.); 4·92 in. (say 5 in.).
11. (*a*) $\frac{7}{8}$ in. diameter bolts, 1, 2 and 3 respectively; (*b*) 1 in. diameter bolts, 1, 2 and 3 respectively. (1 in. bolts could be used throughout.)
12. 10 in. deep, 6 in. wide; approx. $\frac{1}{500}$; 56 lbs. per sq. in.
13. (*a*) 4450 lbs. per sq. in. (Note. This is rather a high value); (*b*) 1,523,800 lbs. per sq. in.; (*c*) 6533 lbs. per sq. in.
14. 35,555 lbs.; $F = 8$ nearly.

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